

Derivatives

Definition

The *derivative* of a function f is a new function defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

We will say that a function f is *differentiable* over an interval (a, b) if the derivative function $f'(x)$ at every point in (a, b) .

Q. How is this the same or different from what we were doing yesterday with tangent lines?

A. Yesterday, we were calculating derivatives at individual points, and getting *numbers* for answers. Today, we'll calculate the *derivative function*, and get out answers with variables in them (do all the points at once).

Example: let $f(x) = x^2$

Derivatives at a point: If I first ask “what is $f'(2)$?”, I could calculate

$$f'(2) = \lim_{h \rightarrow 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h \rightarrow 0} \frac{4h + h^2}{h} = \lim_{h \rightarrow 0} 4 + h = \boxed{4}.$$

But then, if I ask “what is $f'(3)$?” we have to do it all over again.

Today's goal: Write down a *function* $f'(x)$ which has all the derivatives-at-a-point collected together.

If a is a number, (like 2 or 3) then

$$f'(a) = \lim_{h \rightarrow 0} \underbrace{\frac{f(a+h) - f(a)}{h}}_{\substack{\text{a function of} \\ \text{a's and h's}}} = \text{number}$$

gets rid of the h's

But x is a variable, so

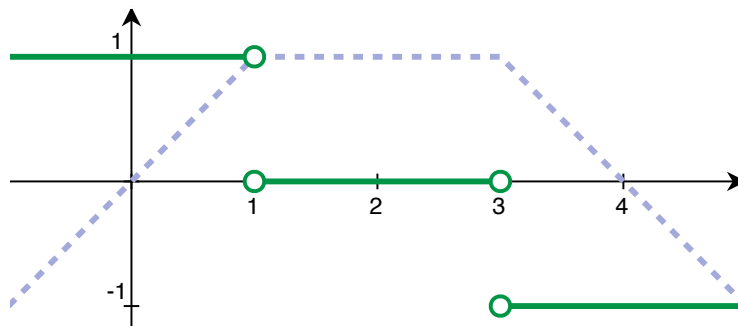
$$f'(x) = \lim_{h \rightarrow 0} \underbrace{\frac{f(x+h) - f(x)}{h}}_{\substack{\text{a function of} \\ \text{x's and h's}}} = \text{function of x's}$$

gets rid of the h's

Starting simple

Suppose we consider the piecewise linear function

$$f(x) = \begin{cases} x & x \leq 1 \\ 1 & 1 < x < 3 \\ -x + 4 & 3 \leq x \end{cases}$$



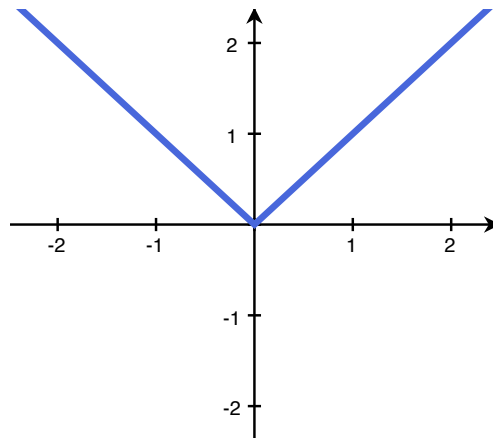
The derivative is:

$$f'(x) = \begin{cases} 1 & x < 1 \\ 0 & 1 < x < 3 \\ -1 & 3 < x \end{cases}$$

Another example

What is the derivative of $f(x) = |x|$?

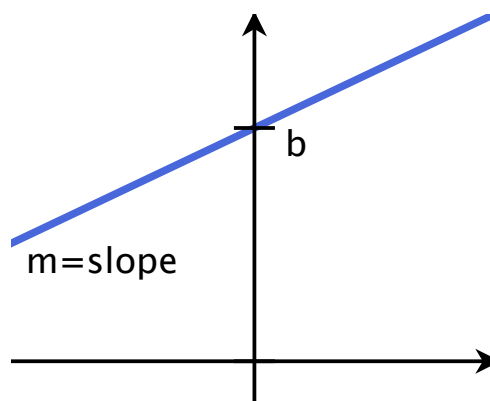
Write down the piecewise function and sketch it on the graph.



Lines

In general, if m and b are constants, and

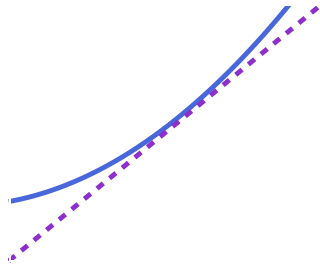
$$f(x) = mx + b$$



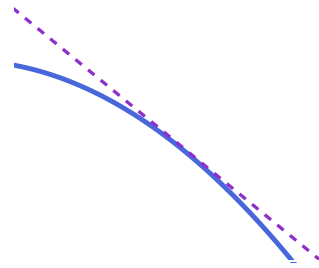
$$f'(x) = m$$

the slope of the tangent line = slope of the line

Rough shape of the derivative



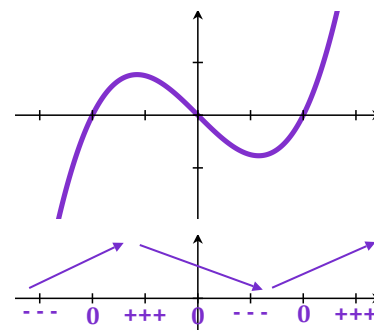
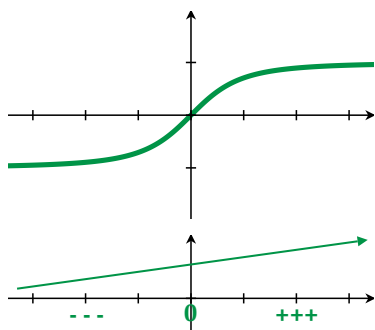
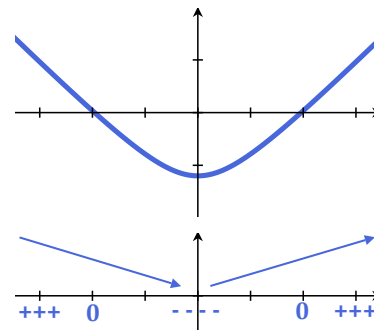
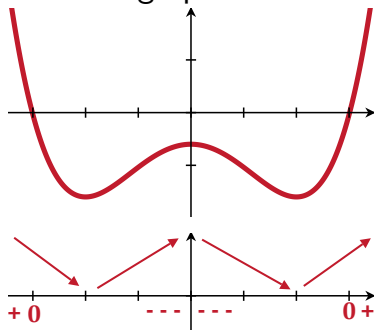
if $f(x)$ is increasing
 $f'(x)$ is positive!



if $f(x)$ is decreasing
 $f'(x)$ is negative!

Match em up!

Here are graphs of two functions and their derivatives. Which are which?



A little more notation

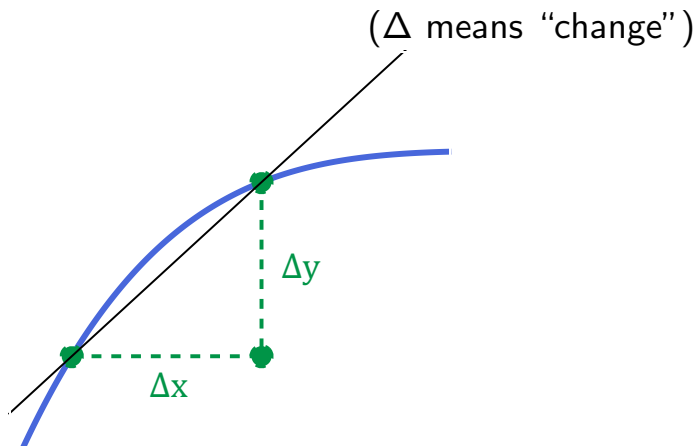
Back to what $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ means:

Rename

$$h = \Delta x$$

and

$$f(x+h) - f(x) = \Delta y$$



$$\text{So } m = \frac{f(x+h) - f(x)}{h} = \frac{\Delta y}{\Delta x}.$$

A little more notation

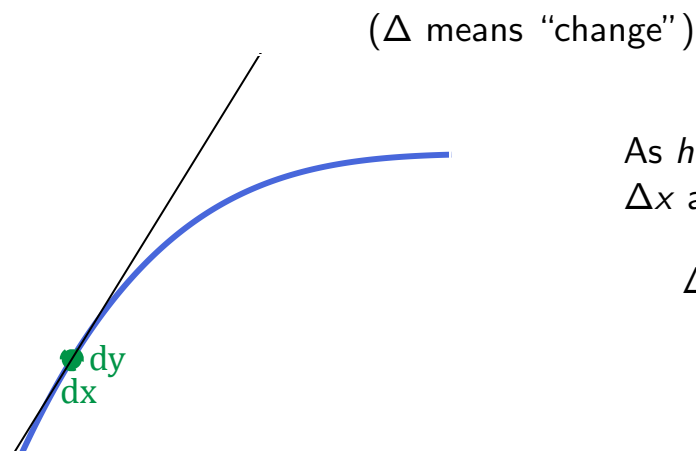
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Rename

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and

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As $h \rightarrow 0$,
 Δx and Δy get infinitely small.

$$\Delta x \rightsquigarrow dx \quad \Delta y \rightsquigarrow dy$$

$$\frac{\Delta x}{\Delta y} \rightsquigarrow \frac{dx}{dy}$$

"infinitesimals"

$$\text{So } m = \frac{f(x+h) - f(x)}{h} = \frac{\Delta y}{\Delta x}.$$

Leibniz notation

One way to write the derivative of $f(x)$ versus x is $f'(x)$.
Another way to write it is

$$f'(x) = \frac{df}{dx} = \frac{d}{dx}f(x).$$

Derivatives at a point: $f'(a)$ means the derivative of $f(x)$ evaluated at a . Another way to write it is

$$f'(a) = \left. \frac{df}{dx} \right|_{x=a} = \left. \frac{d}{dx}f(x) \right|_{x=a}$$

Example: We can write the derivative of x^2 as $\frac{d}{dx}x^2$
and the derivative of x^2 at $x = 5$ as $\left. \frac{d}{dx}x^2 \right|_{x=5}$

Go to work: building our first derivative rule.

Example 1: What is the derivative of x^2 ?

$$\begin{aligned} \frac{d}{dx}x^2 &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - (x)^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} 2x + h = \boxed{2x} \quad \left(\text{so } \left. \frac{d}{dx}x^2 \right|_{x=5} = 2 * 5 \right) \end{aligned}$$

By taking limits, fill in the rest of the table:

$f(x)$	1	x	x^2	x^3	$\frac{1}{x}$	$\frac{1}{x^2}$	\sqrt{x}	$\sqrt[3]{x}$
$f'(x)$			$2x$					

Hints: For $\frac{1}{x^2}$, find a common denominator, and then expand.

For $\sqrt[3]{x}$, try multiplying and dividing by $(\sqrt[3]{x+h})^2 - (\sqrt[3]{x+h})(\sqrt[3]{x}) + (\sqrt[3]{x})^2$.

$f(x)$	$f'(x)$
$x^0 = 1$	0
$x^1 = x$	1
x^2	$2x$
x^3	$3x^2$
$\frac{1}{x} = x^{-1}$	$-x^{-2}$
$\frac{1}{x^2} = x^{-2}$	$-2x^{-3}$
$\sqrt{x} = x^{1/2}$	$(1/2)x^{-1/2}$
$\sqrt[3]{x} = x^{1/3}$	$(1/3)x^{-2/3}$

Power rule:

$$\frac{d}{dx}x^a = ax^{a-1}$$

Use the power rule to take consecutive derivatives of $x^{5/2}$:

$$\begin{aligned}
 x^{5/2} &\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2}x^{3/2}} \quad 1^{\text{st}} \text{ derivative} = f'(x) = \frac{d}{dx}x^2 \\
 &\xrightarrow{\frac{d}{dx}} \boxed{\phantom{\frac{5}{2}x^{3/2}}} \quad 2^{\text{nd}} \text{ derivative} = f''(x) = \frac{d^2}{dx^2}x^2 \\
 &\xrightarrow{\frac{d}{dx}} \boxed{\phantom{\frac{5}{2}x^{3/2}}} \quad 3^{\text{rd}} \text{ derivative} = f^{(3)}(x) = \frac{d^3}{dx^3}x^2 \\
 &\xrightarrow{\frac{d}{dx}} \boxed{\phantom{\frac{5}{2}x^{3/2}}} \\
 &\quad 4^{\text{th}} \text{ derivative} = f^{(4)}(x) = \frac{d^4}{dx^4}x^2
 \end{aligned}$$

Definition: The n^{th} derivative of $f(x)$ is

$$\underbrace{\frac{d}{dx} \frac{d}{dx} \dots \frac{d}{dx}}_n f(x) = \frac{d^n}{dx^n} f(x) = f^{(n)}(x).$$