Derivatives

Definition

The *derivative* of a function f is a new function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We will say that a function f is *differentiable* over an interval (a, b) if if the derivative function $f'(x)$ at every point in (a, b).

Q. How is this the same or different from what we were doing yesterday with tangent lines?

A. Yesterday, we were calculating derivatives at individual points, and getting *numbers* for answers. Today, we'll calculate the *derivative function*, and get out answers with variables in them (do all the points at once).

Example: let $f(x) = x^2$

Derivatives at a point: If I first ask "what is $f'(2)$?", I could calculate

$$f'(2) = \lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h\to 0} \frac{4h + h^2}{h} = \lim_{h\to 0} 4 + h = \boxed{4}.$$

But then, if I ask "what is $f'(3)$?" we have to do it all over again.

Today's goal: Write down a *function* $f'(x)$ which has all the derivatives-at-a-point collected together.

If a is a number, (like 2 or 3) then

$$f'(a) = \underbrace{\lim_{h \to 0}}_{\substack{\text{gets rid} \\ \text{of the } h\text{'s}}} \underbrace{\frac{f(a+h) - f(a)}{h}}_{\substack{\text{a function of} \\ a\text{'s and } h\text{'s}}} = \text{number}$$

But x is a variable, so

$$f'(x) = \underbrace{\lim_{h \to 0}}_{\substack{\text{gets rid} \\ \text{of the } h\text{'s}}} \underbrace{\frac{f(x+h) - f(x)}{h}}_{\substack{\text{a function of} \\ x\text{'s and } h\text{'s}}} = \text{function of } x\text{'s}$$

Starting simple

Suppose we consider the piecewise linear function

$$f(x) = \begin{cases} x & x \leq 1 \\ 1 & 1 < x < 3 \\ -x + 4 & 3 \leq x \end{cases}$$

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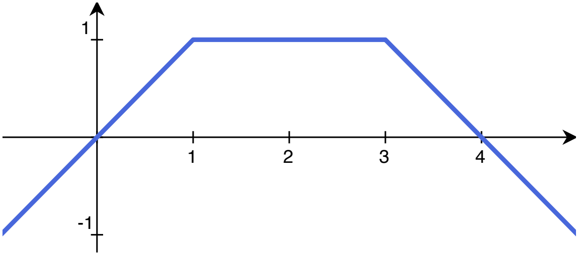

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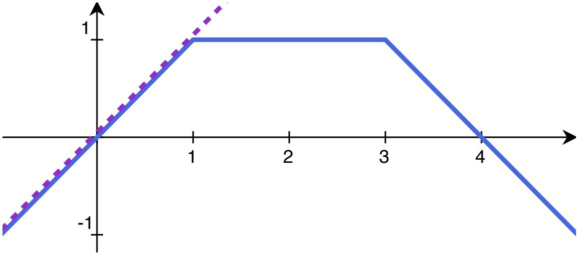

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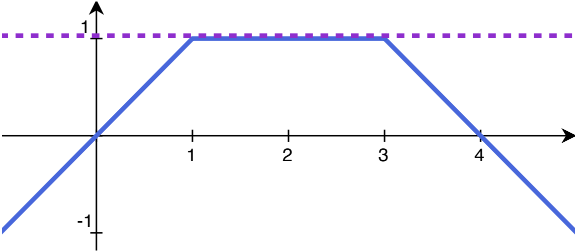

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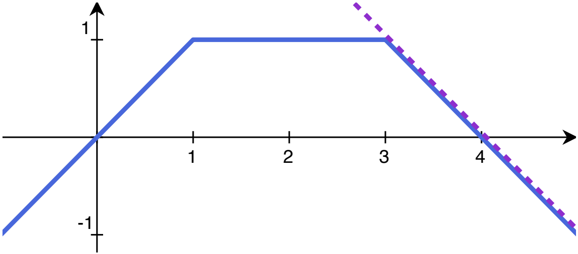

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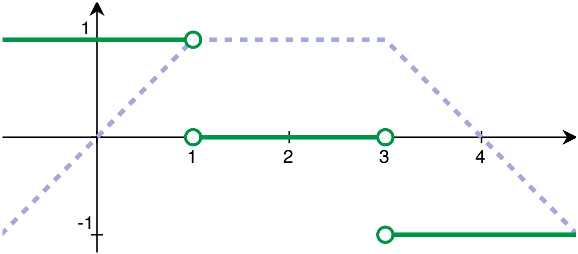


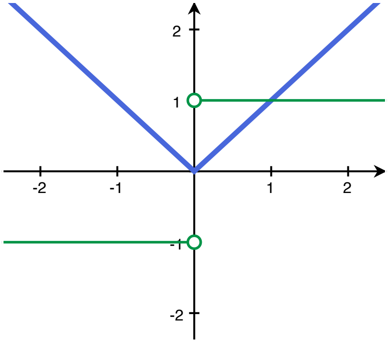
The derivative is:

$$f(x) = \begin{cases} 1 & x < 1 \\ 0 & 1 < x < 3 \\ -1 & 3 < x \end{cases}$$

Another example

What is the derivative of $f(x) = |x|$?

Write down the piecewise function and sketch it on the graph.

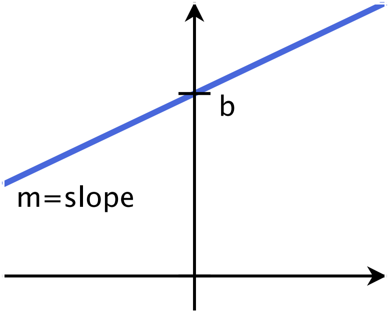


$$f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$

Lines

In general, if m and b are constants, and

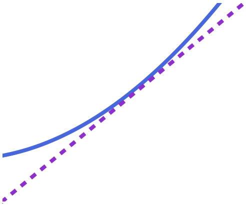
$$f(x) = mx + b$$



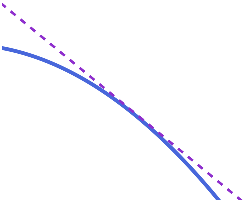
$$f'(x) = m$$

the slope of the tangent line = slope of the line

Rough shape of the derivative



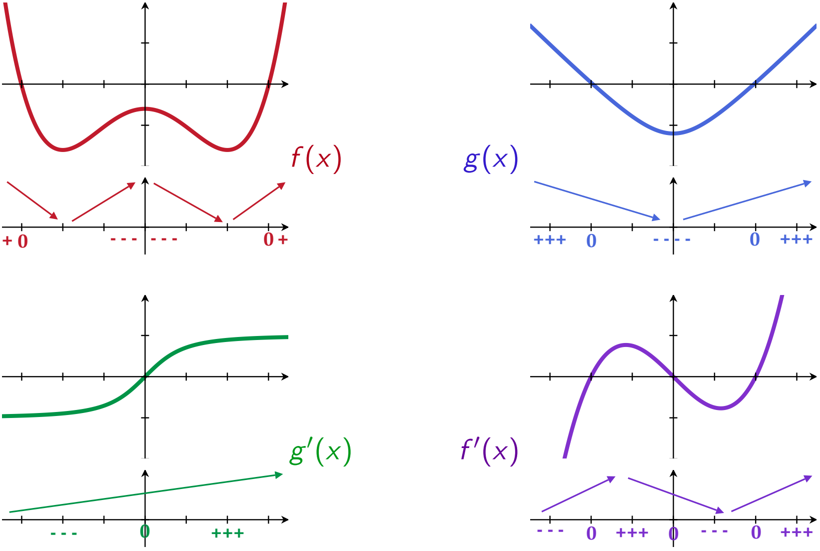
if $f(x)$ is increasing
$f'(x)$ is positive!



if $f(x)$ is decreasing
$f'(x)$ is negative!

Match em up!

Here are graphs of two functions and their derivatives. Which are which?

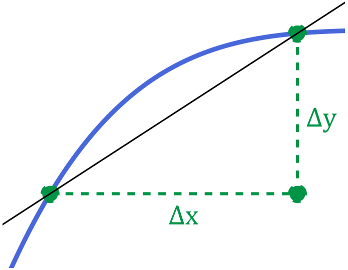


A little more notation

Back to what $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ means:

Rename $\boxed{h = \Delta x}$ and $\boxed{f(x+h) - f(x) = \Delta y}$

(Δ means "change")



So $m = \frac{f(x+h)-f(x)}{h} = \frac{\Delta y}{\Delta x}$.

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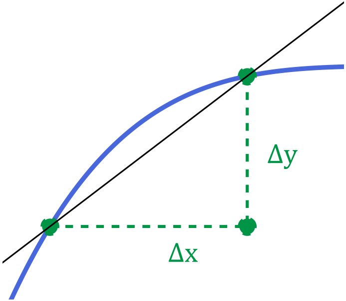

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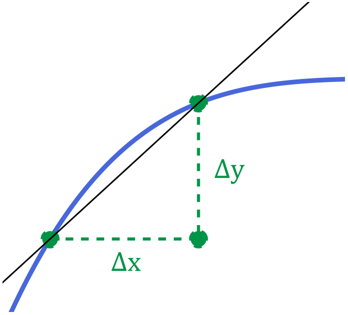

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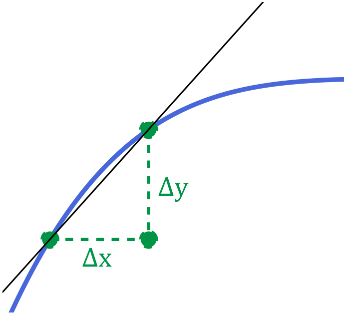

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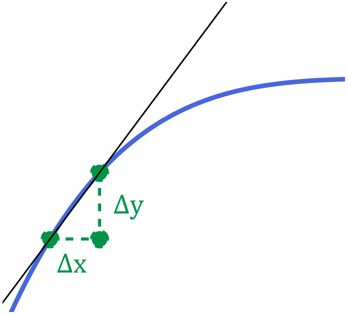

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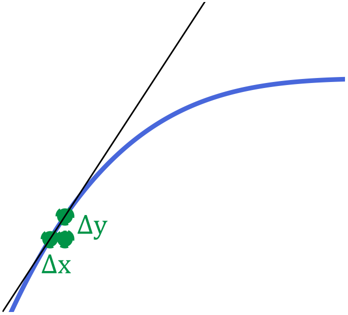

As $h \to 0$,
Δx and Δy get infinitely small.

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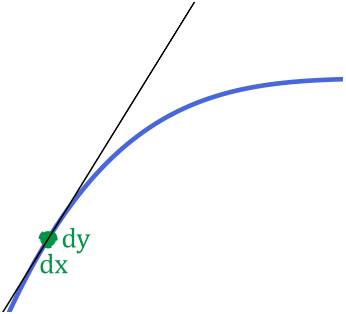

As $h \to 0$,
Δx and Δy get infinitely small.

$$\Delta x \rightsquigarrow dx \qquad \Delta y \rightsquigarrow dy$$

$$\frac{\Delta y}{\Delta x} \rightsquigarrow \frac{dy}{dx}$$

"infinitesimals"

So $m = \frac{f(x+h)-f(x)}{h} = \frac{\Delta y}{\Delta x}$.

Leibniz notation

One way to write the derivative of $f(x)$ versus x is $f'(x)$. Another way to write it is

$$f'(x) = \frac{df}{dx} = \frac{d}{dx}f(x).$$

Derivatives at a point: $f'(a)$ means the derivative of $f(x)$ *evaluated at* a. Another way to write it is

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}$$

Example: We can write the derivative of x^2 as $\frac{d}{dx}x^2$

and the derivative of x^2 at $x = 5$ as $\left.\frac{d}{dx}x^2\right|_{x=5}$

Go to work: building our first derivative rule.

Example 1: What *is* the derivative of x^2?

$$\frac{d}{dx}x^2 = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-(x)^2}{h}$$

$$= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h}$$

$$= \lim_{h\to 0} 2x+h = \boxed{2x} \qquad \left(\text{so } \frac{d}{dx}x^2\Big|_{x=5} = 2*5\right)$$

By taking limits, fill in the rest of the table:

$f(x)$	1	x	x^2	x^3	$\frac{1}{x}$	$\frac{1}{x^2}$	$\sqrt{x}$	$\sqrt[3]{x}$
$f'(x)$	0	1	$2x$	$3x^2$	$-\frac{1}{x^2}$	$-\frac{2}{x^3}$	$\frac{1}{2\sqrt{x}}$	$\frac{1}{3(\sqrt[3]{x})^2}$

Hints: For $\frac{1}{x^2}$, find a common denominator, and then expand.

For $\sqrt[3]{x}$, try multiplying and dividing by $(\sqrt[3]{x+h})^2 + (\sqrt[3]{x+h})(\sqrt[3]{x}) + (\sqrt[3]{x})^2$.

$f(x) = x$

$$\lim_{h \to 0} \frac{(x+h) - x}{h} \stackrel{*}{=} \lim_{h \to 0} \frac{h}{h} = \boxed{1}$$

$f(x) = x^2$

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$\stackrel{*}{=} \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} 2x + h$$

$$= \boxed{2x}$$

$f(x) = x^3$

recall: $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

so $(x+h)^3 - x^3 = 3x^2h + 3xh^2 + h^3$.

So, if $h \neq 0$

$$* \quad \frac{(x+h)^3 - x^3}{h} = 3x^2 + 3xh + h^2$$

So

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \left(3x^2 + 3xh + h^3\right)$$

$$= \boxed{3x^2} + 0 + 0$$

$$f(x) = \frac{1}{x} = x^{-1}$$

$$\lim_{h\to 0} \frac{\left(\frac{1}{x+h} - \frac{1}{x}\right)}{h} = \lim_{h\to 0}\left(\frac{1}{h}\right)\left(\frac{x-(x+h)}{(x+h)(x)}\right)$$

$$\overset{*}{=} \lim_{h\to 0}\left(\frac{1}{h}\right)\left(\frac{-h}{(x+h)x}\right)$$

$$= \lim_{h\to 0} -\frac{1}{(x+h)x} = -\frac{1}{x^2} = \boxed{-x^{-2}}$$

$$f(x) = \frac{1}{x^2} = x^{-2}$$

$$\lim_{h\to 0} \frac{\left(\frac{1}{(x+h)^2} - \frac{1}{x^2}\right)}{h} = \lim_{h\to 0}\left(\frac{1}{h}\right)\left(\frac{x^2-(x+h)^2}{(x+h)^2x^2}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{x^2-(x^2+2xh+h^2)}{(x+h)^2\ x^2}\right)$$

$$\overset{*}{=} \lim_{h\to 0} \frac{1}{h}\left(\frac{-2xh+h^2}{(x+h)^2x^2}\right) = \lim_{h\to 0} \frac{-2x+h}{(x+h)^2x^2}$$

$$= -\frac{2x}{x^4} = -\frac{2}{x^3} = \boxed{-2x^{-3}}$$

From last class:

$$f(x) = \sqrt{x}$$

so $f(x+h) = \sqrt{x+h}$

so

can't plug in $h=0$ yet

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \quad \begin{matrix}\to 0 \\ \to 0\end{matrix}$$

$$= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

so

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2x^{1/2}}$$

$f(x) = \sqrt[3]{x} = x^{1/3}$:

Since

$$(a-b)(a^2 + ab + b^2)$$

$$= a^3 + a^2 b + ab^2$$
$$- (a^2 b + ab^2 + b^3)$$

$$= a^3 - b^3 ,$$

let $a = \sqrt[3]{x+h}$
$b = \sqrt[3]{x}$

we have

$$\left((x+h)^{1/3} - x^{1/3}\right)\left((x+h)^{2/3} + (x+h)^{1/3} x^{1/3} + x^{2/3}\right)$$

$$= (x+h)^{3/3} - x^{3/3} = x + h - x = h .$$

So

$$\lim_{h \to 0} \frac{(x+h)^{1/3} - x^{1/3}}{h} \cdot \frac{\left((x+h)^{2/3} + (x+h)^{1/3} x^{1/3} + x^{2/3}\right)}{\left((x+h)^{2/3} + (x+h)^{1/3} x^{1/3} + x^{2/3}\right)}$$

*

$$= \lim_{h \to 0} \frac{h}{h\left((x+h)^{2/3} + (x+h)^{1/3} x^{1/3} + x^{2/3}\right)}$$

$$= \frac{1}{x^{2/3} + x^{1/3} x^{1/3} + x^{2/3}} = \frac{1}{3x^{2/3}} = \boxed{\frac{1}{3} x^{-2/3}}$$

$f(x)$	$f'(x)$
$x^0 = 1$	0
$x^1 = x$	1
x^2	$2x$
x^3	$3x^2$
$\frac{1}{x} = x^{-1}$	$-x^{-2}$
$\frac{1}{x^2} = x^{-2}$	$-2x^{-3}$
$\sqrt{x} = x^{1/2}$	$(1/2)x^{-1/2}$
$\sqrt[3]{x} = x^{1/3}$	$(1/3)x^{-2/3}$

Power rule:

$$\frac{d}{dx}x^a = ax^{a-1}$$

Use the power rule to take consecutive derivatives of $x^{5/2}$:

$$x^{5/2} \xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2}x^{3/2}} \quad \textit{1}^{st} \textit{ derivative} = f'(x) = \frac{d}{dx}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2}x^{1/2}} \quad \textit{2}^{nd} \textit{ derivative} = f''(x) = \frac{d^2}{dx^2}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2} * \frac{1}{2}x^{-1/2}} \quad \textit{3}^{rd} \textit{ derivative} = f^{(3)}(x) = \frac{d^3}{dx^3}x^2$$

$$\xrightarrow{\frac{d}{dx}} \boxed{\frac{5}{2} * \frac{3}{2} * \frac{1}{2} * \left(-\frac{1}{2}\right) x^{-3/2}}$$

$$\textit{4}^{th} \textit{ derivative} = f^{(4)}(x) = \frac{d^4}{dx^4}x^2$$

Definition: The n^{th} derivative of $f(x)$ is

$$\underbrace{\frac{d}{dx}\frac{d}{dx}\dots\frac{d}{dx}}_{n} f(x) = \frac{d^n}{dx^n}f(x) = f^{(n)}(x).$$