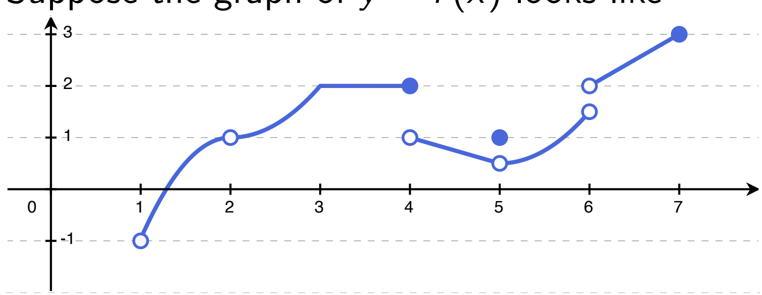
Continuity

Warm-up

Suppose the graph of $y = f(x)$ looks like

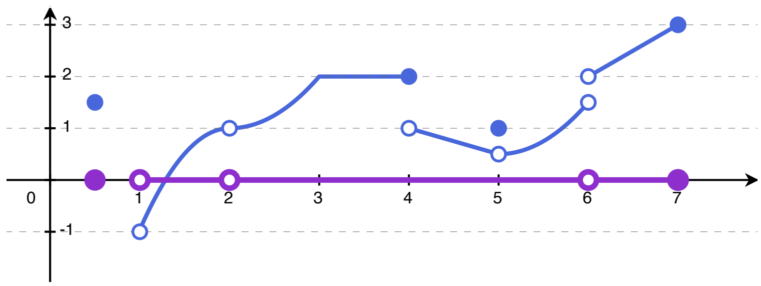


answers to 4.:

a	$x \to a^-$	$x \to a^+$

1. What is the domain of $f(x)$?
2. What is the range of $f(x)$?
3. For which values a in $[1, 7]$ does $\lim_{x \to a} f(x)$ not exist?
4. For those values you picked out in 3., what are
$$\lim_{x \to a^-} f(x) \qquad \text{and} \qquad \lim_{x \to a^+} f(x)?$$
5. Which values a satisfy
$$f(a) \text{ and } \lim_{x \to a} f(x) \text{ exist, but } f(a) \neq \lim_{x \to a} f(x)?$$

Domain definitions



Let D be the domain of $f(x)$. Ex. $D = (1,2) \cup (2,6) \cup (6,7]$

Ex 2. $D = \{\frac{1}{2}\} \cup (1,2) \cup (2,6) \cup (6,7]$

Definition

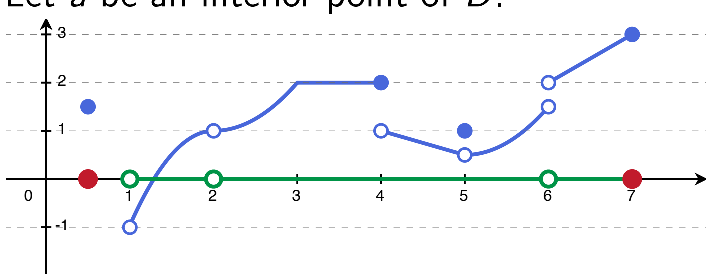
An *interior point* of D is any point **in** D which is not an endpoint or an isolated point.

Ex. Everything in D except $x = 7$.

Ex 2. Everything in D except $x = \frac{1}{2}$ & 7.

Continuity

Let a be an interior point of D.



Ex. $f(x)$ is discontinuous at $x = 4$ and 5.
No other points are fair game!

Definition

A function is *continuous* at a if $\lim_{x \to a} f(x) = f(a)$. If it is not continuous at a, then function is *discontinuous* at a.

Checklist:

1. Is a an interior point? *If no, stop here... we'll get back to these.*
2. Does (a) $\lim\limits_{x \to a^-} f(x)$ exist? (b) $\lim\limits_{x \to a^+} f(x)$ exist?
3. Does $\lim\limits_{x \to a} f(x)$ exist? (i.e. does (a) = (b)?)
4. Does $f(a) = \lim\limits_{x \to a} f(x)$?

If the answer to any of 2.–4. is "no", then $f(x)$ is discontinuous at a.

Some examples:

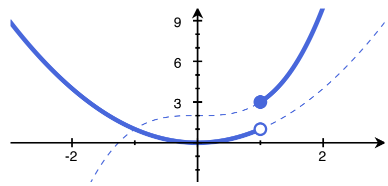
Over their domains, all polynomials, rational functions, trigonometric functions, exponential functions, absolute values, and their inverses are all continuous functions. (Jumps all happen over domain gaps)

Example: Is the function $f(x) = \begin{cases} x^2 & x < 1 \\ x^3 + 2 & 1 \leq x \end{cases}$ continuous?

Solution: The only possible problem would happen at $x = 1$. Let's check there:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^3 + 2 = 3$$



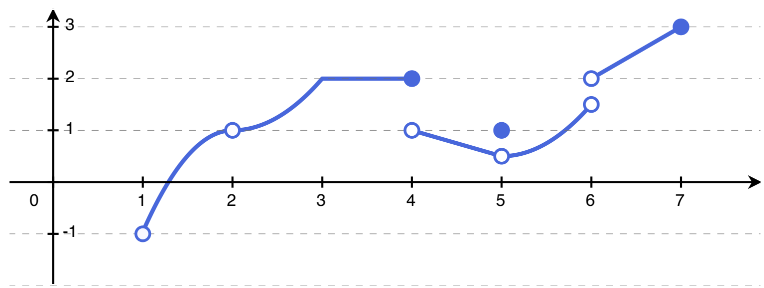
$\boxed{\text{No}}$, $f(x)$ is discontinuous at $x = 1$ because 1 is an interior point of the domain, but $\lim_{x \to 1} f(x)$ does not exist.

Right Continuity and Left Continuity

Definition

A function $f(x)$ is *right continuous* at a point a if it is defined on an interval $[a, b)$ and $\lim_{x\to a^+} f(x) = f(a)$.
Similarly, a function $f(x)$ is *left continuous* at a point a if it is defined on an interval $(b, a]$ and $\lim_{x\to a^-} f(x) = f(a)$.

Example:



$f(x)$ is

(a) continuous at every *interior* point in D *except* $x = 4$ and 5;

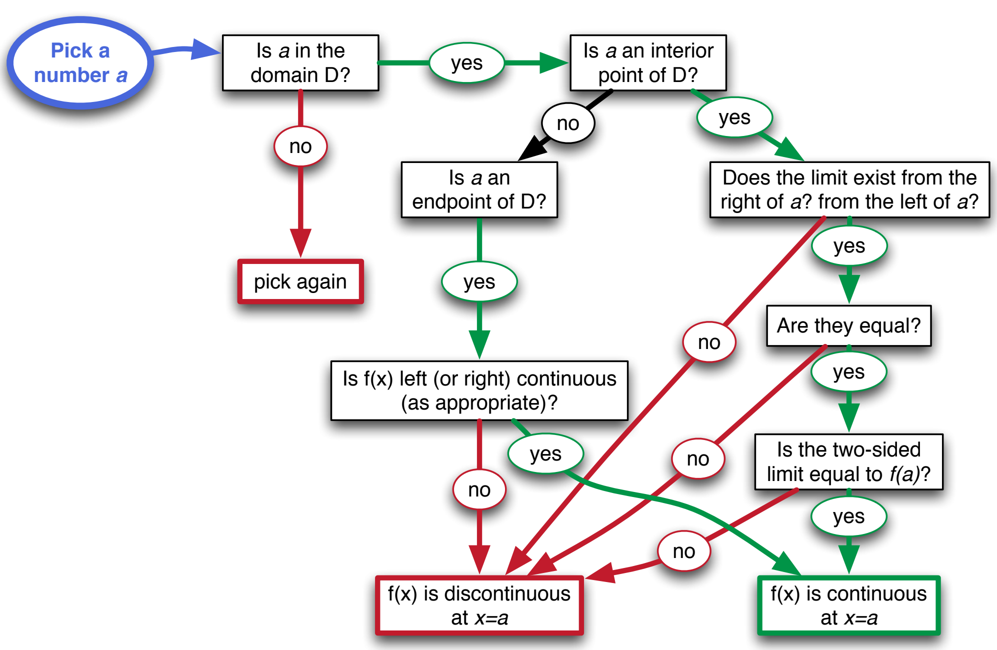
(b) only right continuous at those points included in (a); and

(c) additionally left continuous at $x = 4$ and $x = 7$.

Suppose a function f has no isolated points in its domain.

Definition

A function f is *continuous over its domain D* if **(1)** is is continuous at every interior point of D, and **(2)** it is left (or right) continuous at every endpoint of D. Otherwise, it has a *discontinuity* at each point in D which violates (1) or (2).



Filling and Fixing

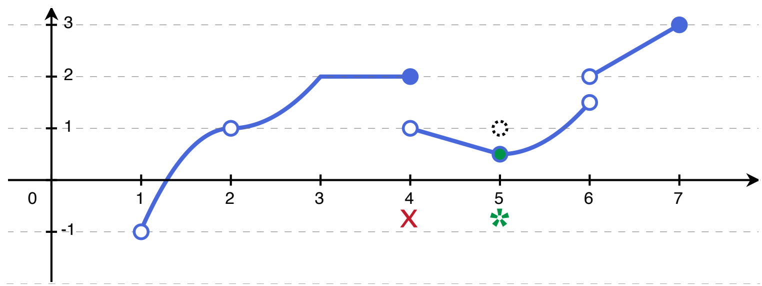
Suppose a is a point of discontinuity in D

(a) If a is an interior point and $\lim_{x\to a} f(x) = L$ exists; or

(b) if a is an endpoint and $\lim_{x\to a^\pm} f(x) = L$ exists,

then we say $f(x)$ has a *removable discontinuity*:

$$\bar{f}(x) = \begin{cases} f(x) & x \neq a \\ L & x = a \end{cases}$$



Example: $f(x)$ has a removable discontinuity in exactly one place:

$$\bar{f}(x) = \begin{cases} f(x) & x \neq 5 \\ 1 & x = 5 \end{cases}$$

Filling and Fixing

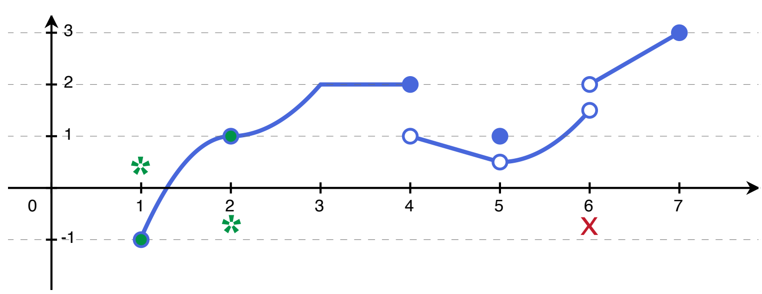
Suppose a is a hole in D (a is arbitrarily close to points in D, but not in D).

(a) If a would be an interior point and $\lim_{x\to a} f(x) = L$ exists; or

(b) if a would be an endpoint and $\lim_{x\to a^\pm} f(x) = L$ exists,

then we say $f(x)$ has a *continuous extension*:

$$\bar{f}(x) = \begin{cases} f(x) & x \neq a \\ L & x = a \end{cases}$$



Example: $f(x)$ has continuous extensions in exactly two places:

$$\bar{f}_1(x) = \begin{cases} f(x) & x \neq 1 \\ -1 & x = 1 \end{cases} \quad \text{and} \quad \bar{f}_2(x) = \begin{cases} f(x) & x \neq 2 \\ 1 & x = 2 \end{cases}$$

Examples

(A) Which of the following have removable discontinuities? For those which do, what are the alternate functions with those discontinuities removed?

(B) Which of the following have continuous extensions? For those which do, what are those extensions?

1. $f(x) = \dfrac{x^2 - 4}{x - 2}$

2. $f(x) = \begin{cases} \sin x & x \neq \pi/3 \\ 0 & x = \pi/3 \end{cases}$

3. $f(x) = \dfrac{|x|}{x}$

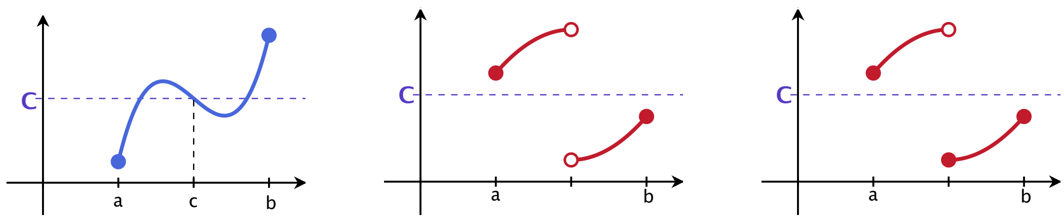
One application: The Intermediate Value Theorem

Suppose f is continuous on a closed interval $[a, b]$.

$$\text{If} \qquad f(a) < C < f(b) \qquad \text{or} \qquad f(a) > C > f(b),$$

then there is at least one point c in the interval $[a, b]$ such that

$$f(c) = C.$$



Example 1: Show that the equation $x^5 - 3x + 1 = 0$ has at least one solution in the interval $[0, 1]$.
Example 2: Show every polynomial

$$p(x) = a_n x^n + \cdots + a_1 x + a_0, \qquad a_n \neq 0$$

of odd degree has at least one real root (a solution to $p(x) = 0$).

Extra practice:

Where is a function continuous?

In general: What does it mean for a function $f(x)$ to be continuous at $x = a$? Explain how to test if a function is continuous at $x = a$.

Specifically:

1. For which values of x is the function $f(x) = x^2 + 3x + 4$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

2. For which values of x is the function $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & \text{if } x \neq 3, \\ 5, & \text{if } x = 3, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

* 3. For which values of x is the function $f(x) = \begin{cases} \dfrac{\sin 3x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

* 4. For which values of x is the function $f(x) = \begin{cases} \dfrac{1 - \cos x}{x^2}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

* 5. Determine the value of k for which the function $f(x) = \begin{cases} \dfrac{\sin 2x}{5x}, & \text{if } x \neq 0, \\ k, & \text{if } x = 0, \end{cases}$ is continuous at $x = 0$. Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

6. For which values of x is the function $f(x) = \begin{cases} x - 1, & \text{if } 1 \leq x < 2, \\ 2x - 3, & \text{if } 2 \leq x \leq 3, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

7. For which values of x is the function $f(x) = \begin{cases} \cos x, & \text{if } x \geq 0, \\ -\cos x, & \text{if } x < 0, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

8. For which values of x is the function $f(x) = \begin{cases} \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

***Save #3-5 for later**

9. Find the value of a for which the function $f(x) = \begin{cases} ax+5, & \text{if } x \le 2, \\ x-1, & \text{if } x > 2, \end{cases}$ is continuous at $x = 2$. Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

10. For which values of x is the function $f(x) = \begin{cases} 1+x^2, & \text{if } 0 \le x \le 1, \\ 2-x, & \text{if } x > 1, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

11. For which values of x is the function $f(x) = 2x - |x|$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

12. Find the value of a for which the function $f(x) = \begin{cases} 2x-1, & \text{if } x < 2, \\ a, & \text{if } x = 2, \\ x+1, & \text{if } x > 2, \end{cases}$ is continuous at $x = 2$. Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

13. For which values of x is the function $f(x) = \begin{cases} \dfrac{|x-a|}{x-a}, & \text{if } x \ne a, \\ 1, & \text{if } x = a, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

14. For which values of x is the function $f(x) = \begin{cases} \dfrac{x-|x|}{2}, & \text{if } x \ne 0, \\ 2, & \text{if } x = 0, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

15. For which values of x is the function $f(x) = \begin{cases} \sin x, & \text{if } x < 0, \\ x, & \text{if } x \ge 0, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

16. For which values of x is the function $f(x) = \begin{cases} \dfrac{x^n-1}{x-1}, & \text{if } x \ne 1, \\ n, & \text{if } x = 1, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

17. Explain how you know that $f(x) = \sec x$ is continuous for all values of x. Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

18. For which values of x is the function $f(x) = \cos|x|$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

19. For which values of x is the function $f(x) = \lfloor x \rfloor$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

20. For which values of x is the function $f(x) = \begin{cases} x^3 - x^2 + 2x - 2, & \text{if } x \neq 1, \\ 4, & \text{if } x = 1, \end{cases}$ continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

21. For which values of x is the function $f(x) = |x| + |x - 1|$, $-1 \leq x \leq 2$, continuous? Justify your answer with limits if necessary and draw a graph of the function to illustrate your answer.

Answers

1. all x	2. all x	3. $x \neq 0$	4. $x \neq 0$
5. $k = 2/4$	6. $1 \leq x \leq 3$	7. $x \neq 0$	8. $x \neq 0$
9. $a = -2$	10 .$x \geq 0,\ x \neq 1$	11. all x	12. $a = 3$
13. $x \neq a$	14. $x \neq 0$	15. all x	16. all x
17.	18. all x	19. x not and integer	20. $x \neq 1$
21. $-1 \leq x \leq 2$			