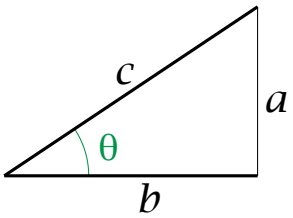
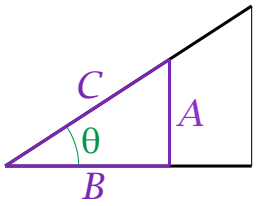
# Trigonometric functions

## Step one: similar triangles




Two similar triangles have the same set of angles, and have the properties that

$$\frac{A}{B} = \frac{a}{b}, \quad \frac{B}{C} = \frac{b}{c}, \text{ and } \frac{A}{C} = \frac{a}{c}.$$

Define

$$\cos(\theta) = \frac{b}{c} \quad \text{and} \quad \sin(\theta) = \frac{a}{c}.$$

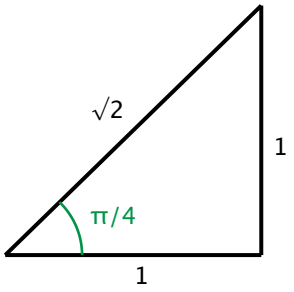
Then let

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{a}{b}, \qquad \sec(\theta) = \frac{1}{\sin(\theta)} = \frac{c}{a},$$

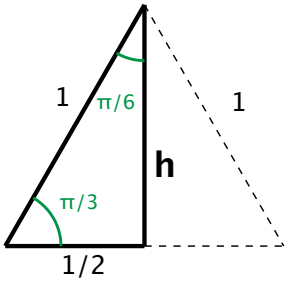
$$\csc(\theta) = \frac{1}{\cos(\theta)} = \frac{c}{b}, \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{b}{a}.$$

## Easy angles:

isosceles right triangle:



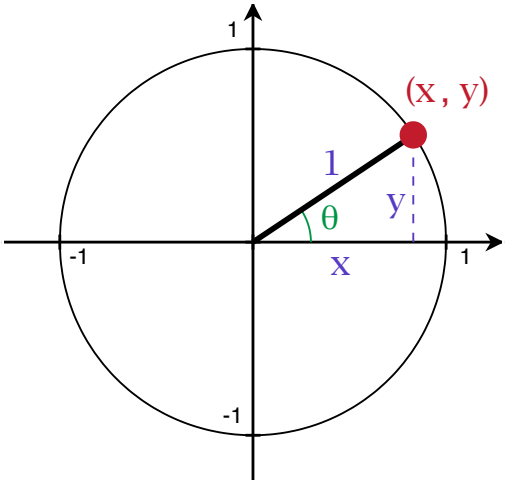

equilateral triangle cut in half:




$h = \sqrt{1-(1/2)^2} = \sqrt{3}/2$

| | $\cos(\theta)$ | $\sin(\theta)$ | $\tan(\theta)$ | $\sec(\theta)$ | $\csc(\theta)$ | $\cot(\theta)$ |
|---|---|---|---|---|---|---|
| $\pi/4$ | | | | | | |
| $\pi/3$ | | | | | | |
| $\pi/6$ | | | | | | |

## Step two: the unit circle




For $0 < \theta < \frac{\pi}{2}$...

$$\cos(\theta) = \frac{x}{1} = x$$

$$\sin(\theta) = \frac{y}{1} = y$$

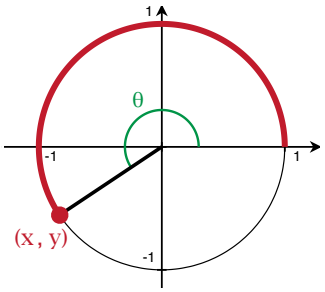
Use this idea to extend trig functions to any $\theta$...

Define

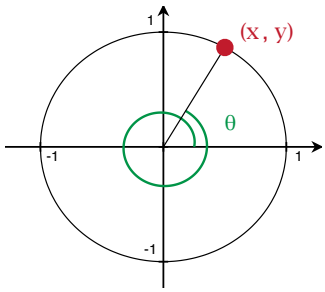
$$\cos(\theta) = x \qquad \sin(\theta) = y,$$

there $\theta$ is defined by...

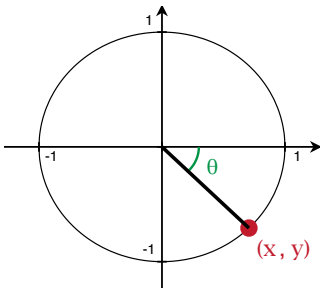
$0 \leq \theta \leq 2\pi$




all $\theta \geq 0$



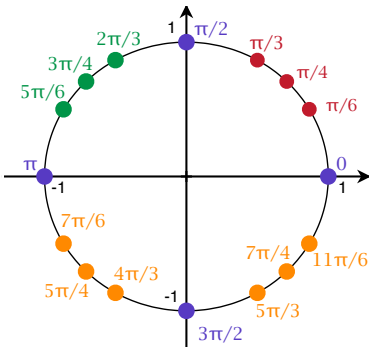

$\theta < 0$




*Sidebar: In calculus, radians are king. Where do they come from?*
*Circumference of a unit circle:* $2\pi$
*Arclength of a wedge with angle* $\theta$*:*

$\frac{\theta}{180^\circ} * 2\pi$ *(if in degrees)* *or* $\frac{\theta}{2\pi} * 2\pi = \boxed{\theta}$ *(if in radians)*

## Reading off of the unit circle




$\cos(\pi - \theta) = -\cos(\theta) \quad \sin(\pi - \theta) = \sin(\theta)$

$\cos(-\theta) = \cos(\theta) \quad \sin(-\theta) = -\sin(\theta)$

$\cos(2\pi n + \theta) = \cos(\theta) \quad \sin(2\pi n + \theta) = \sin(\theta)$

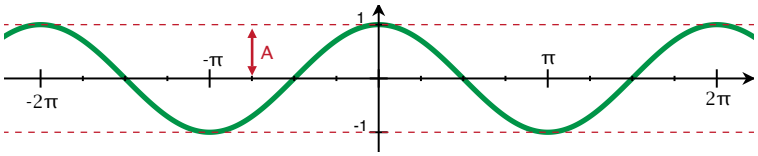
$x^2 + y^2 = 1 \implies \cos^2(\theta) + \sin^2(\theta) = 1$

| | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ |
|---|---|---|---|---|---|---|---|
| $\cos(\theta)$ | | | | | | | |
| $\sin(\theta)$ | | | | | | | |

| | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\frac{7\pi}{6}$ | $\frac{5\pi}{4}$ | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | $\frac{7\pi}{4}$ | $\frac{11\pi}{6}$ |
|---|---|---|---|---|---|---|---|---|---|
| $\cos(\theta)$ | | | | | | | | | |
| $\sin(\theta)$ | | | | | | | | | |

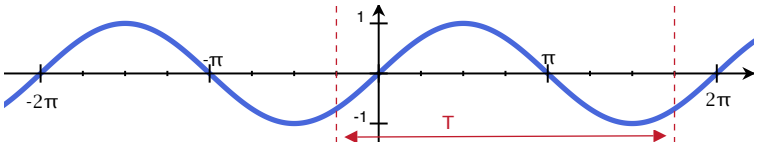
## Plotting on the $\theta$-$y$ axis

Graph of $y = \cos(\theta)$:




$A$ = Amplitude = $\frac{1}{2}$ length of the range = 1
$T$=Period = time to repeat = $2\pi$

Graph of $y = \sin(\theta)$:




$A$=Amplitude = $\frac{1}{2}$ length of the range = 1
$T$=Period = time to repeat = $2\pi$

## Trig identities to know and love:

Even/odd:

$$\cos(-\theta) = \cos(\theta) \quad \text{(even)} \qquad \sin(-\theta) = -\sin(\theta) \quad \text{(odd)}$$

Pythagorean identity:

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

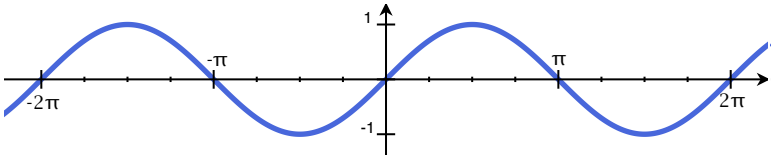
Angle addition:

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$

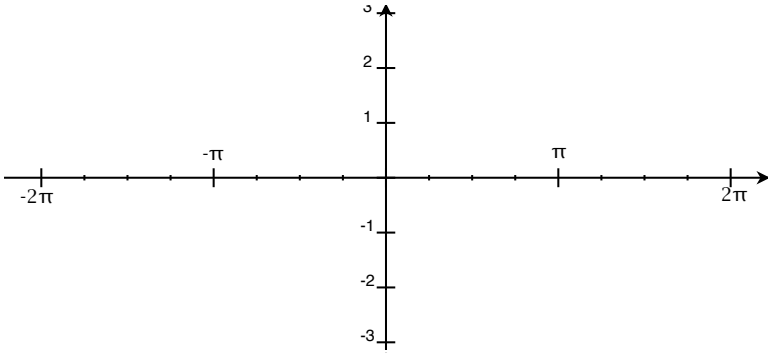
$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$$

*(in particular* $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ *and* $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$*)*

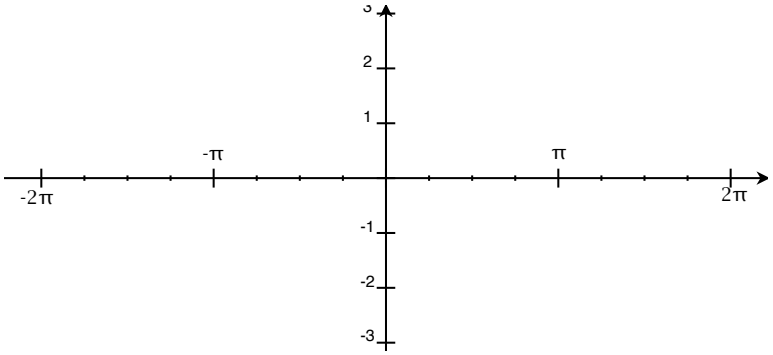
Transform the graph of $\sin(\theta)$ into the graph of $2\sin(\frac{1}{2}\theta + \pi/6) - 1$:



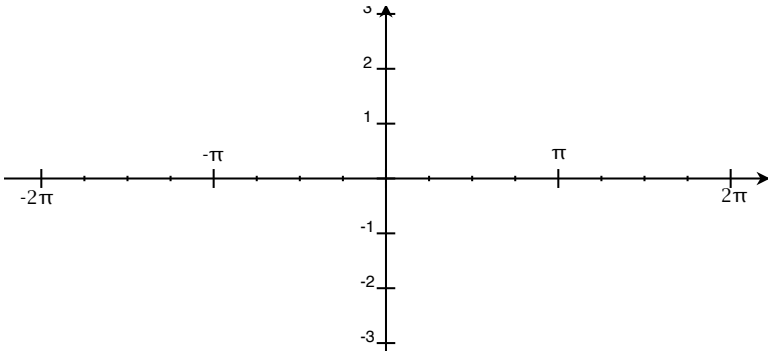

$\sin\left(\frac{1}{2}\theta\right)$ →



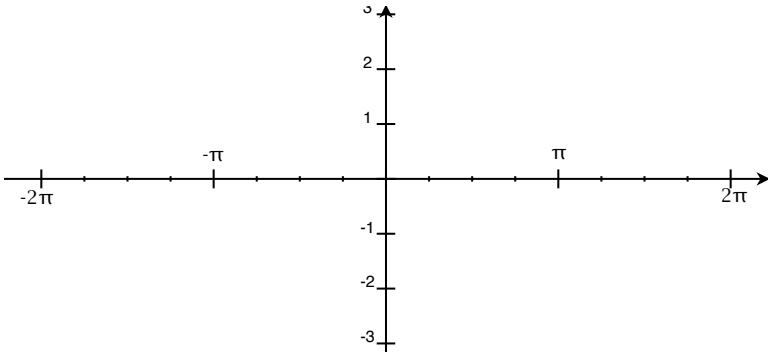

$\sin\left(\frac{1}{2}(\theta+\frac{\pi}{3})\right)$ →




$2\sin\left(\frac{1}{2}(\theta+\frac{\pi}{3})\right)$ →




$2\sin\left(\frac{1}{2}(\theta+\frac{\pi}{3})\right)-1$ →

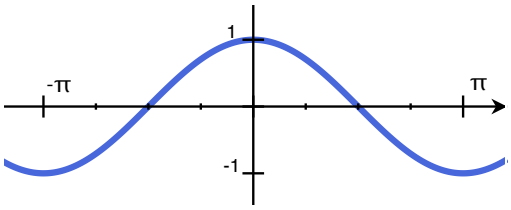



What is the amplitude of $2\sin\left(\frac{1}{2}\theta + \frac{\pi}{6}\right) - 1$?

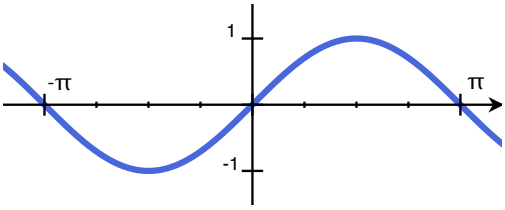
What is the period?

## Other trig functions

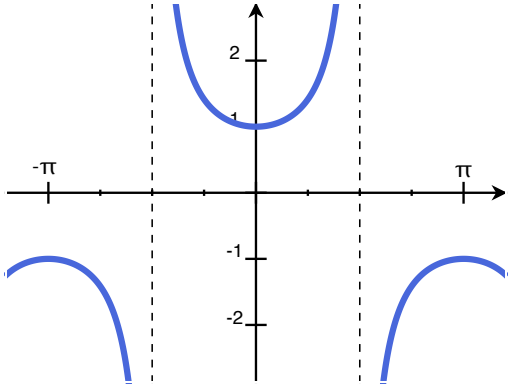
$y = \cos(\theta)$



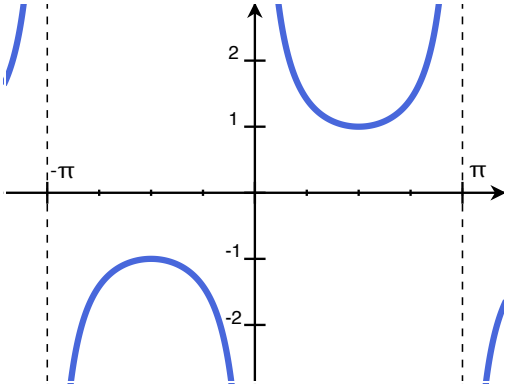

$y = \sin(\theta)$




$\sec(\theta) = 1/\cos(\theta)$

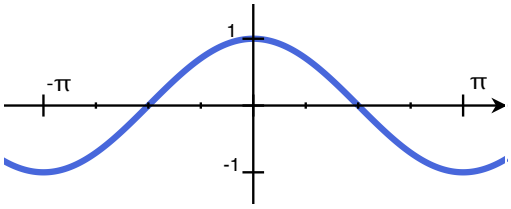



$\csc(\theta) = 1/\sin(\theta)$

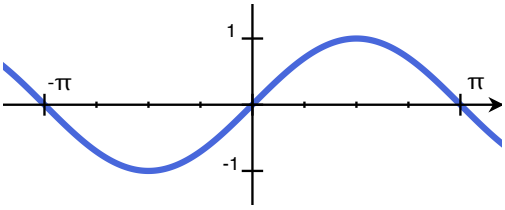



## Other trig functions

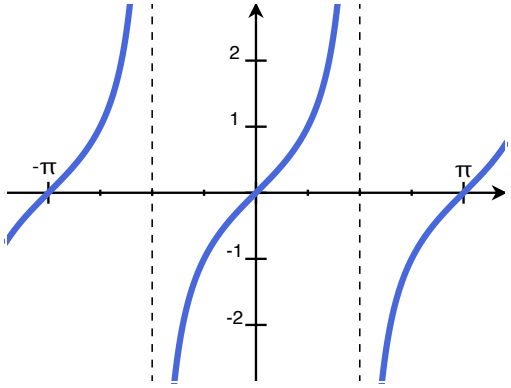
$y = \cos(\theta)$




$y = \sin(\theta)$




$\tan(\theta) = \sin(\theta)/\cos(\theta)$




$\cot(\theta) = \cos(\theta)/\sin(\theta)$