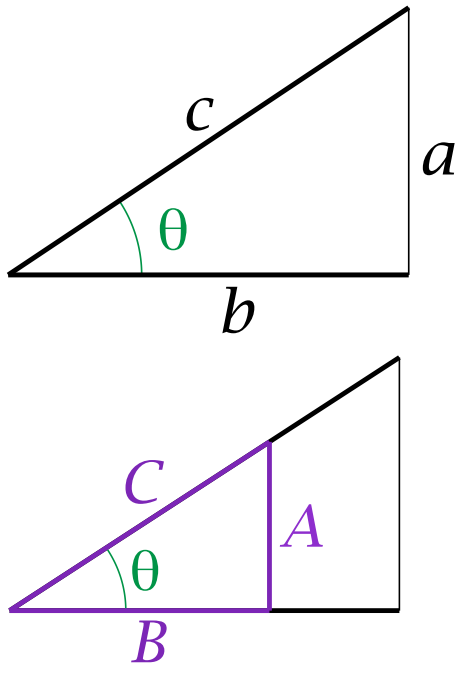
Trigonometric functions

Step one: similar triangles



Two similar triangles have the same set of angles, and have the properties that

$$\frac{A}{B} = \frac{a}{b}, \quad \frac{B}{C} = \frac{b}{c}, \text{ and } \frac{A}{C} = \frac{a}{c}.$$

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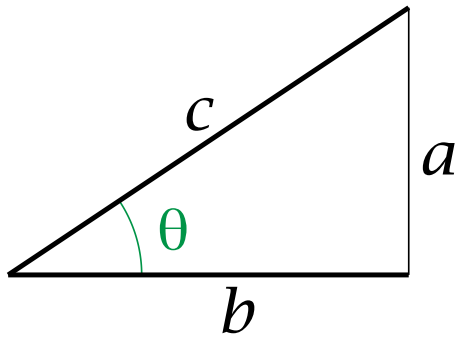

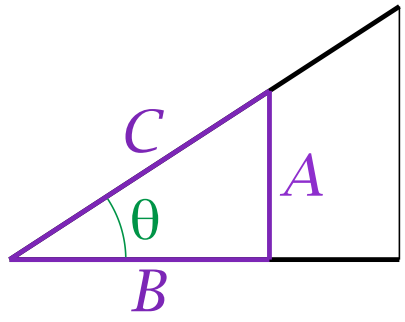

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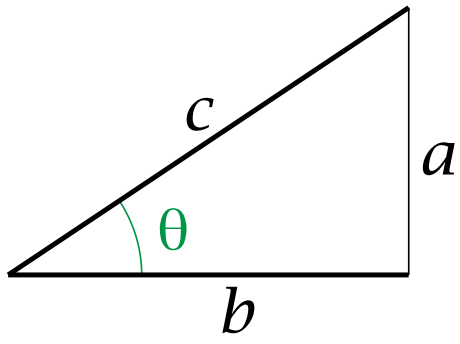
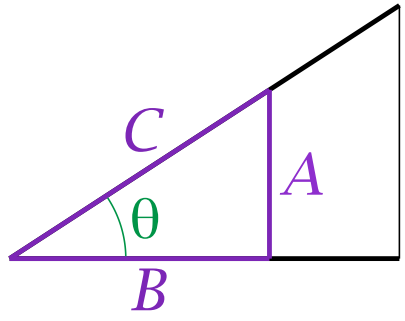
$$\frac{A}{B} = \frac{a}{b}, \quad \frac{B}{C} = \frac{b}{c}, \text{ and } \frac{A}{C} = \frac{a}{c}.$$

Define

$$\cos(\theta) = \frac{b}{c} \quad \text{and} \quad \sin(\theta) = \frac{a}{c}.$$

Step one: similar triangles





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Define

$$\cos(\theta) = \frac{b}{c} \quad \text{and} \quad \sin(\theta) = \frac{a}{c}.$$

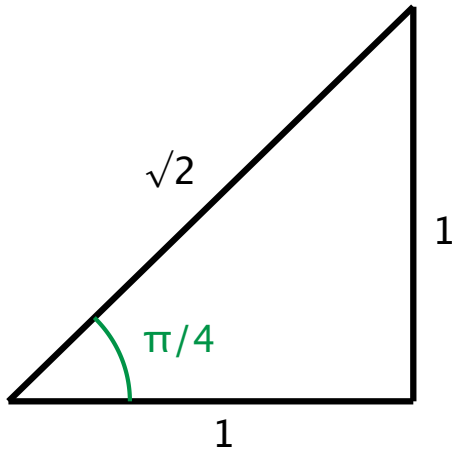
Then let

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{a}{b}, \qquad \sec(\theta) = \frac{1}{\sin(\theta)} = \frac{c}{a},$$

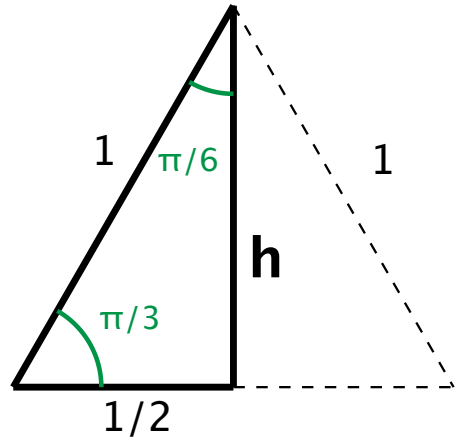
$$\csc(\theta) = \frac{1}{\cos(\theta)} = \frac{c}{b}, \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{b}{a}.$$

Easy angles:

isosceles right triangle:



equilateral triangle cut in half:

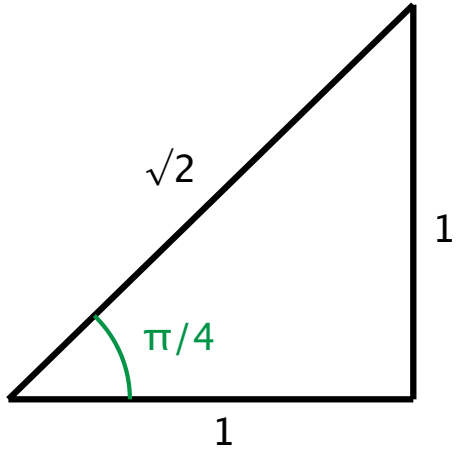


$$h = \sqrt{1 - (1/2)^2} = \sqrt{3}/2$$

	$\cos(\theta)$	$\sin(\theta)$	$\tan(\theta)$	$\sec(\theta)$	$\csc(\theta)$	$\cot(\theta)$
$\pi/4$						
$\pi/3$						
$\pi/6$						

Easy angles:

isosceles right triangle:



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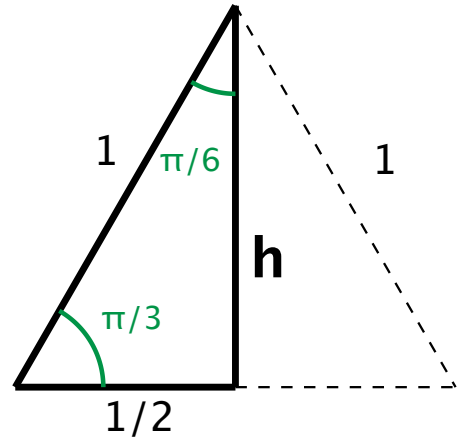
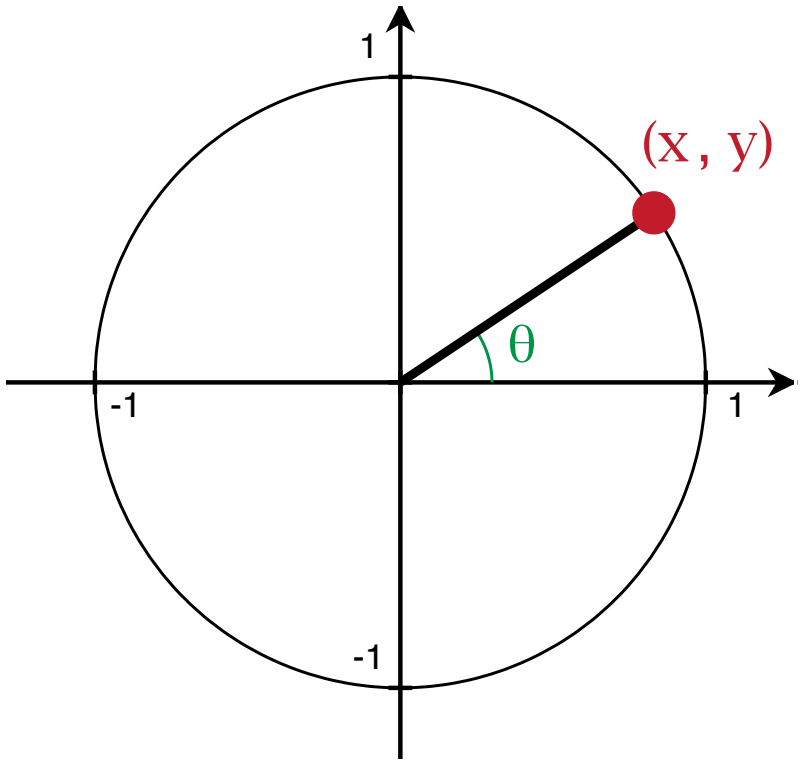


$$h = \sqrt{1-(1/2)^2} = \sqrt{3}/2$$

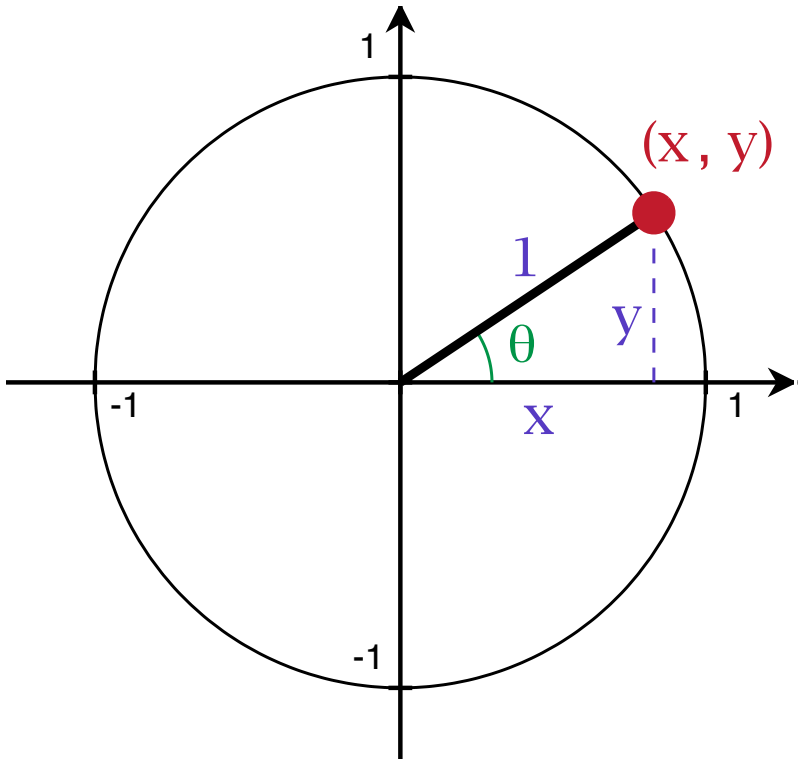
	$\cos(\theta)$	$\sin(\theta)$	$\tan(\theta)$	$\sec(\theta)$	$\csc(\theta)$	$\cot(\theta)$
$\pi/4$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\pi/3$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	2	$\frac{2}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$
$\pi/6$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	2	$\sqrt{3}$

Step two: the unit circle



For $0 < \theta < \frac{\pi}{2}$...

Step two: the unit circle



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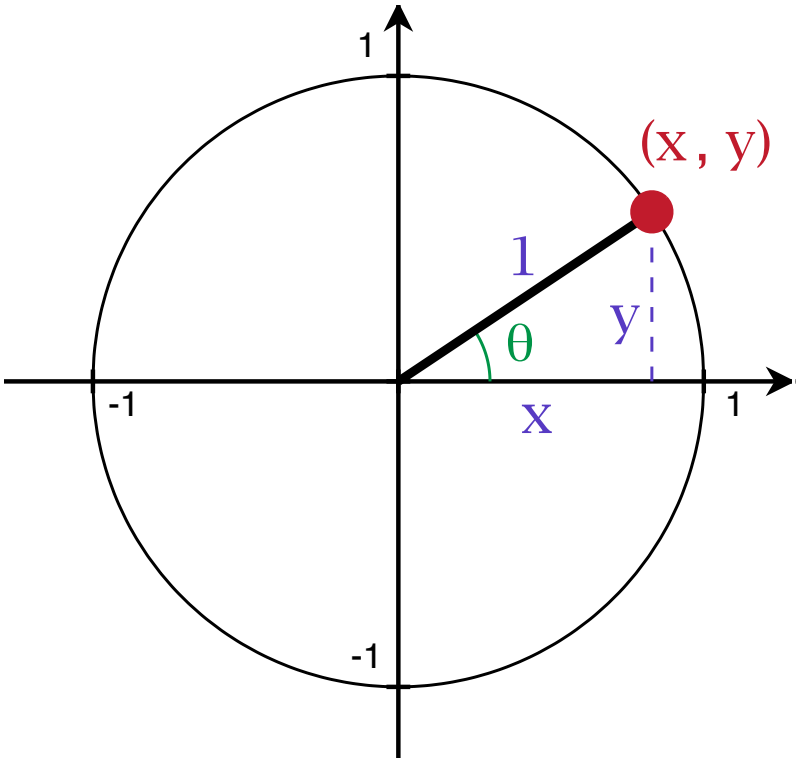


For $0 < \theta < \frac{\pi}{2}$...

$$\cos(\theta) = \frac{x}{1} = x$$

$$\sin(\theta) = \frac{y}{1} = y$$

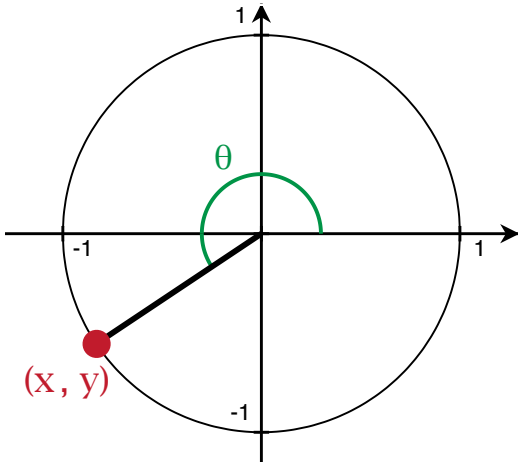
Use this idea to extend trig functions to any θ...

Define

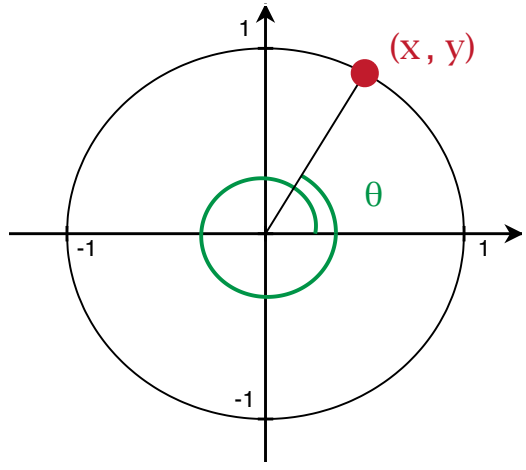
$$\cos(\theta) = x \qquad \sin(\theta) = y,$$

there θ is defined by...

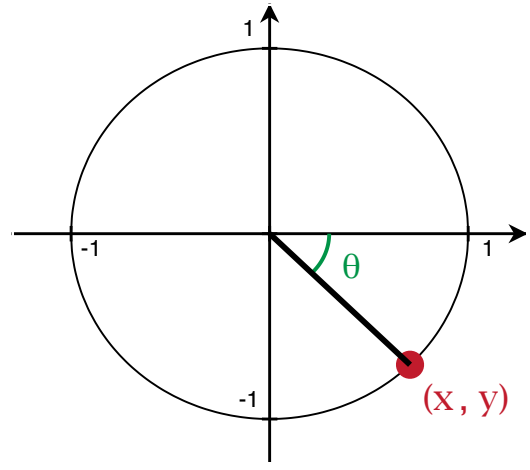
$0 \leq \theta \leq 2\pi$



all $\theta \geq 0$



$\theta < 0$



Define

$$\cos(\theta) = x \qquad \sin(\theta) = y,$$

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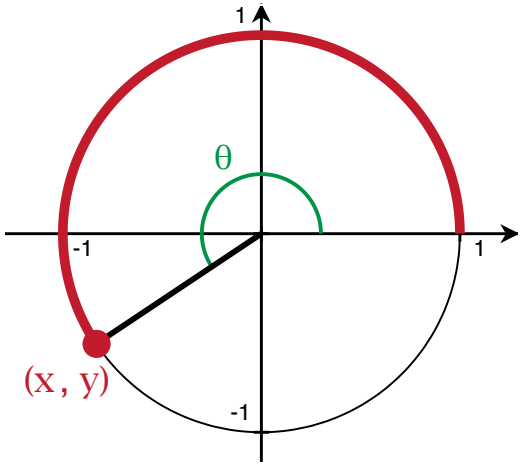
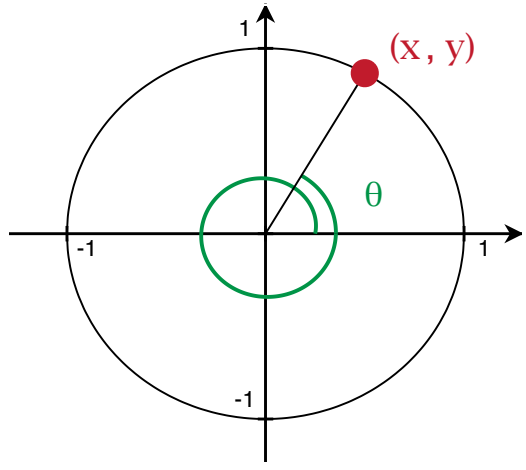
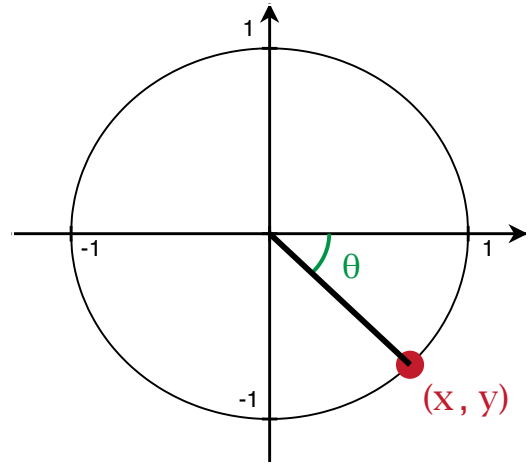


all $\theta \geq 0$



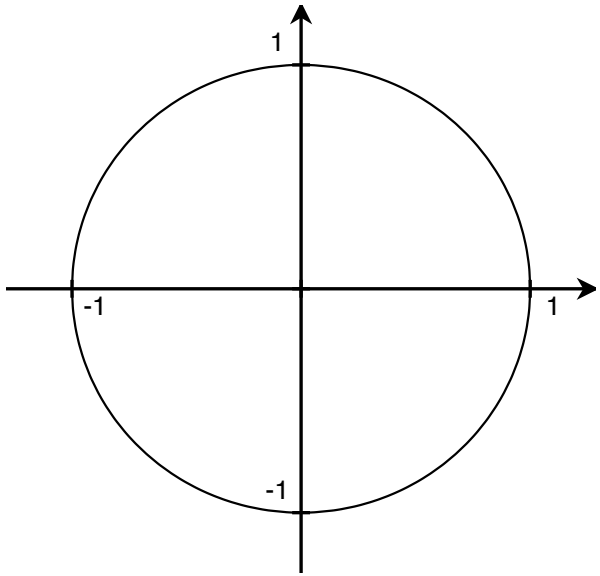
$\theta < 0$



Sidebar: In calculus, radians are king. Where do they come from?
Circumference of a unit circle: 2π
Arclength of a wedge with angle θ*:*

$\frac{\theta}{360^\circ} * 2\pi$ *(if in degrees)* *or* $\frac{\theta}{2\pi} * 2\pi = \boxed{\theta}$ *(if in radians)*

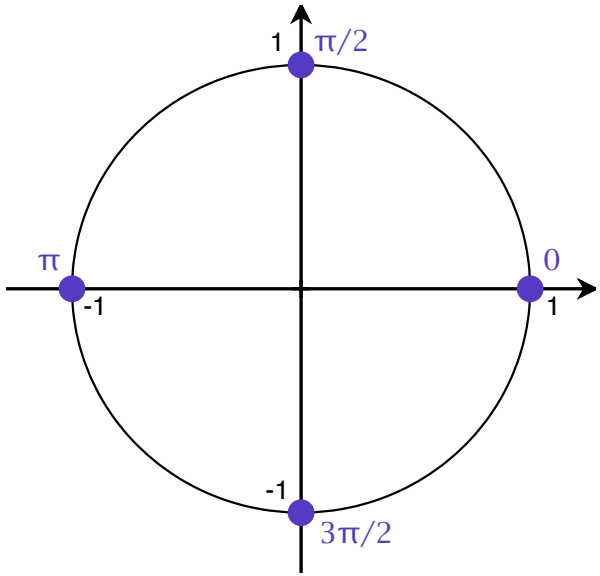
Reading off of the unit circle



	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\cos(\theta)$							
$\sin(\theta)$							

	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$\cos(\theta)$									
$\sin(\theta)$									

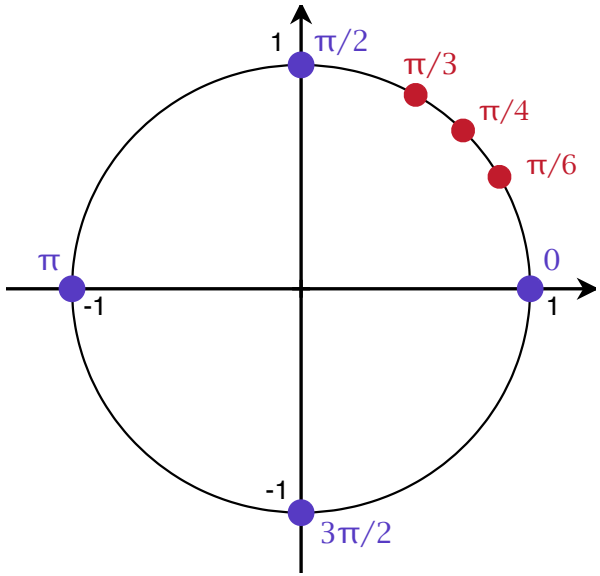
Reading off of the unit circle



	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\cos(\theta)$	1	0	-1	0			
$\sin(\theta)$	0	1	0	-1			

	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$\cos(\theta)$									
$\sin(\theta)$									

Reading off of the unit circle



	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\cos(\theta)$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\sin(\theta)$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$

	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
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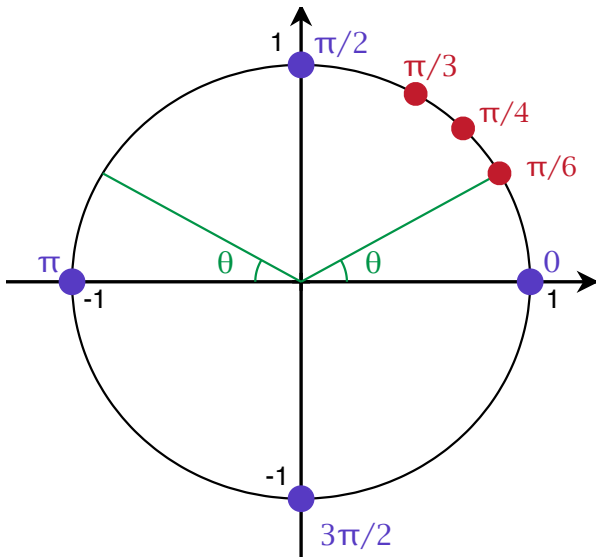
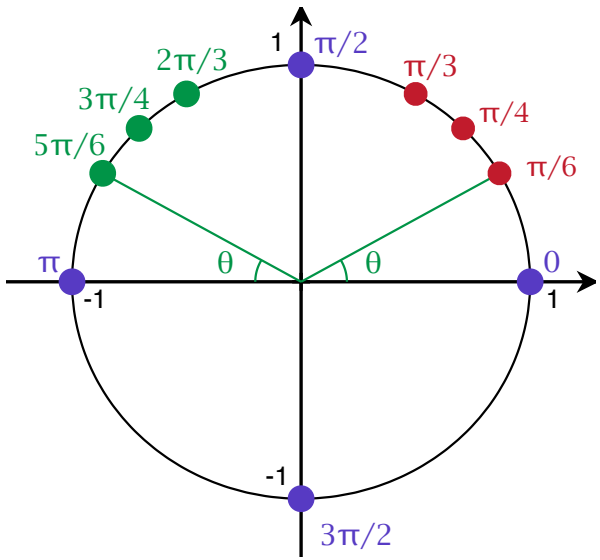


$$\cos(\pi-\theta) = -\cos(\theta) \qquad \sin(\pi-\theta) = \sin(\theta)$$

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\cos(\theta)$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\sin(\theta)$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$

	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$\cos(\theta)$									
$\sin(\theta)$									

Reading off of the unit circle



$$\cos(\pi-\theta) = -\cos(\theta) \qquad \sin(\pi-\theta) = \sin(\theta)$$

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\cos(\theta)$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\sin(\theta)$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$

	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$\cos(\theta)$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$						
$\sin(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$						

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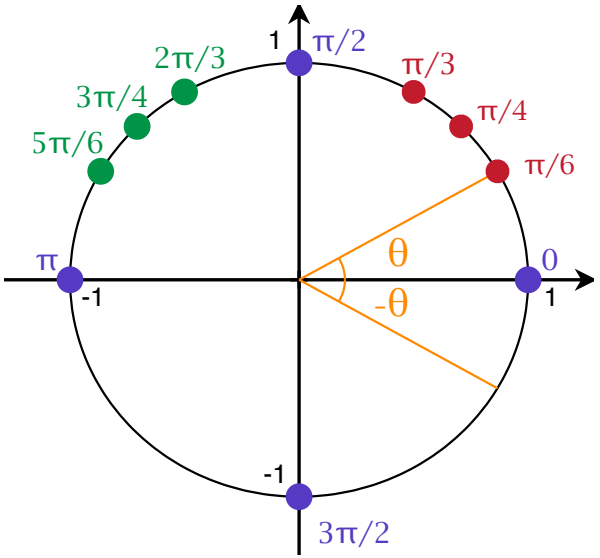


$\cos(\pi-\theta) = -\cos(\theta) \quad \sin(\pi-\theta) = \sin(\theta)$

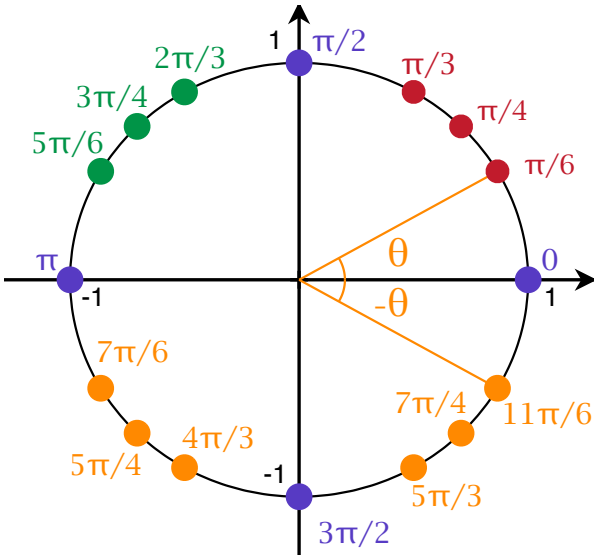
$\cos(-\theta) = \cos(\theta) \quad \sin(-\theta) = -\sin(\theta)$

$\cos(2\pi n+\theta) = \cos(\theta) \quad \sin(2\pi n+\theta) = \sin(\theta)$

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\cos(\theta)$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\sin(\theta)$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$

	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$\cos(\theta)$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$						
$\sin(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$						

Reading off of the unit circle



$$\cos(\pi-\theta) = -\cos(\theta) \quad \sin(\pi-\theta) = \sin(\theta)$$

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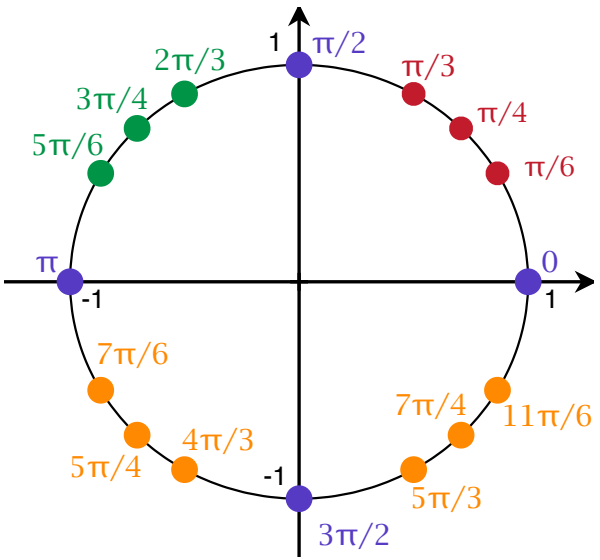
$$\cos(2\pi n+\theta) = \cos(\theta) \quad \sin(2\pi n+\theta) = \sin(\theta)$$

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\cos(\theta)$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\sin(\theta)$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$

	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$\cos(\theta)$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\sin(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$

Reading off of the unit circle



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$$\cos(-\theta) = \cos(\theta) \quad \sin(-\theta) = -\sin(\theta)$$

$$\cos(2\pi n+\theta) = \cos(\theta) \quad \sin(2\pi n+\theta) = \sin(\theta)$$

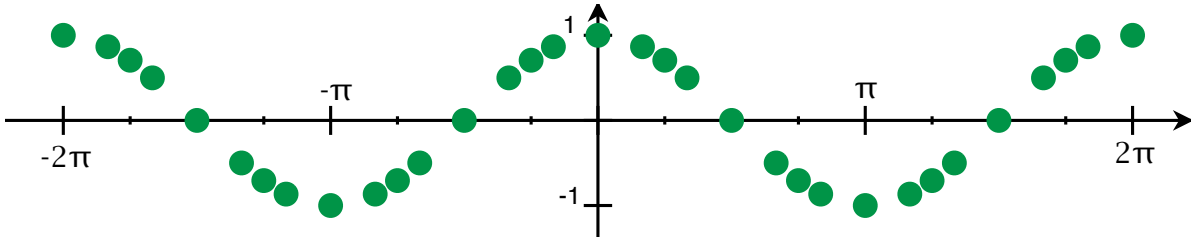
$$x^2+y^2 = 1 \implies \cos^2(\theta)+\sin^2(\theta) = 1$$

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\cos(\theta)$	1	0	-1	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\sin(\theta)$	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$

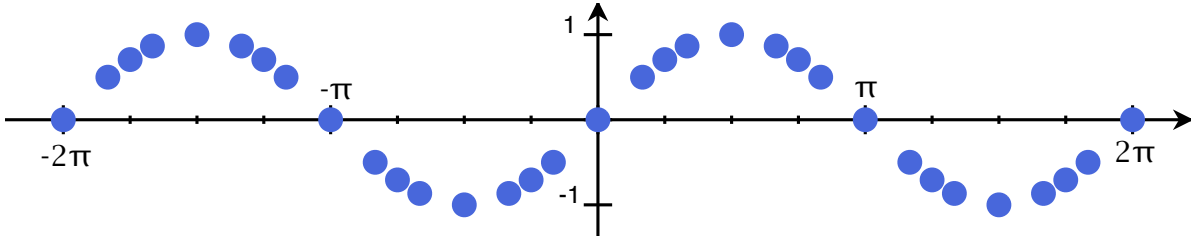
	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$
$\cos(\theta)$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\sin(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$

Plotting on the θ-y axis

Graph of $y = \cos(\theta)$:



Graph of $y = \sin(\theta)$:



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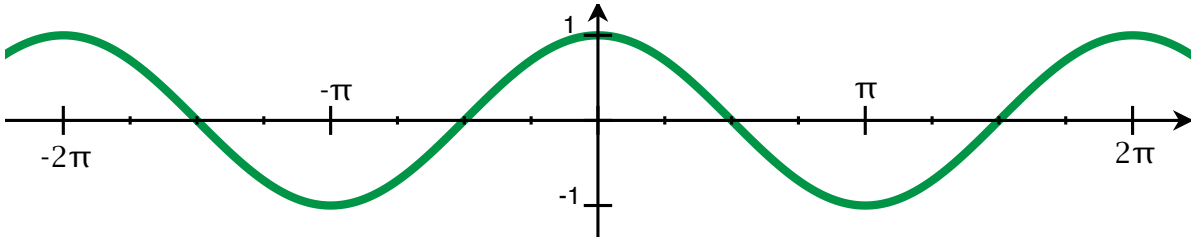
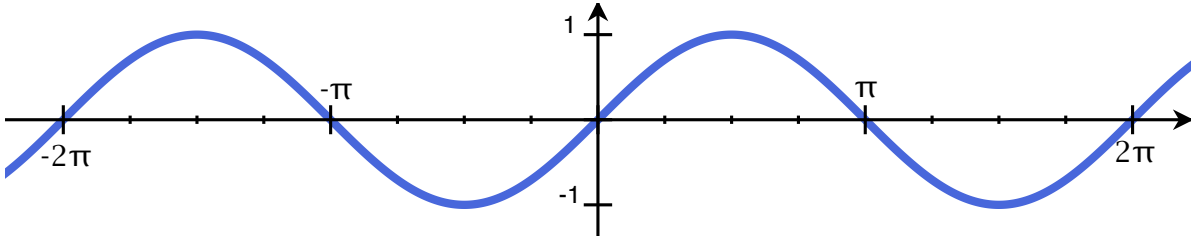
Graph of $y = \cos(\theta)$:

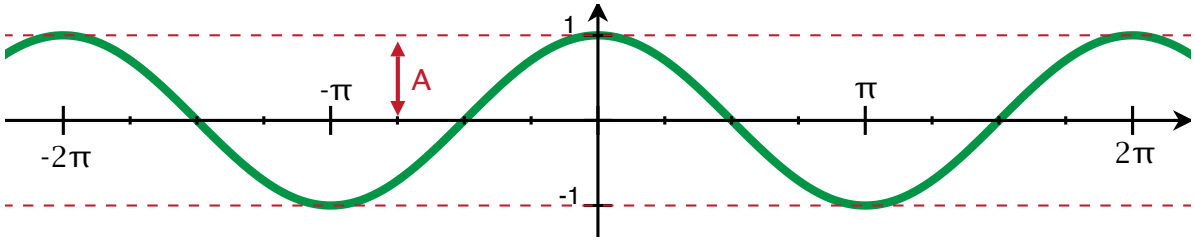


Graph of $y = \sin(\theta)$:



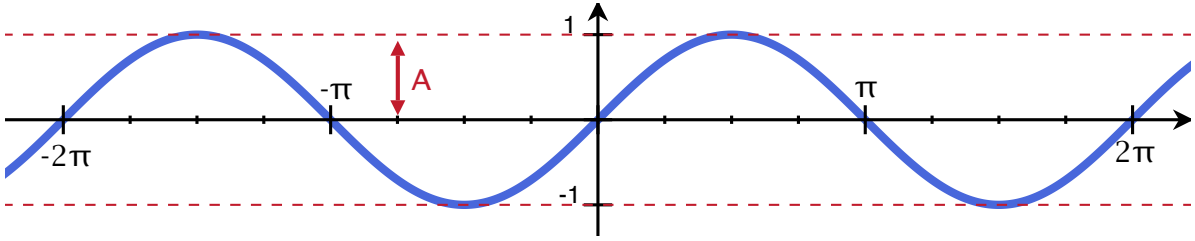
Plotting on the θ-y axis

Graph of $y = \cos(\theta)$:



A = Amplitude = $\frac{1}{2}$ length of the range = 1

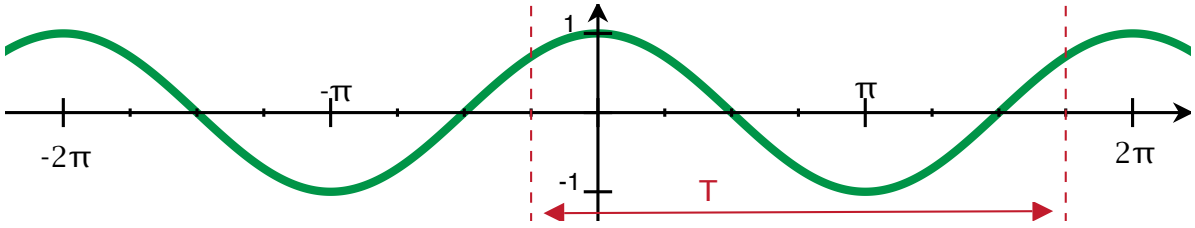
Graph of $y = \sin(\theta)$:



A=Amplitude = $\frac{1}{2}$ length of the range = 1

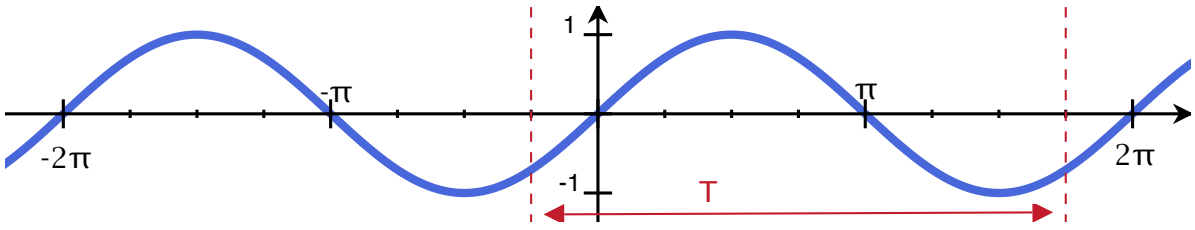
Plotting on the θ-y axis

Graph of $y = \cos(\theta)$:



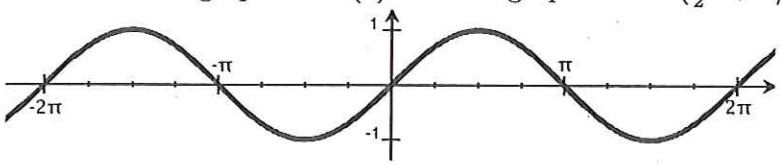
$A =$ Amplitude $= \frac{1}{2}$ length of the range $= 1$
$T =$Period $=$ time to repeat $= 2\pi$

Graph of $y = \sin(\theta)$:



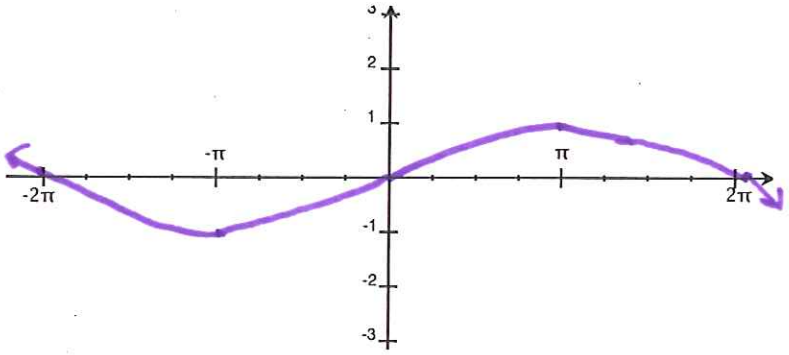
$A =$Amplitude $= \frac{1}{2}$ length of the range $= 1$
$T =$Period $=$ time to repeat $= 2\pi$

Transform the graph of $\sin(\theta)$ into the graph of $2\sin(\frac{1}{2}\theta + \pi/6) - 1$:



$\sin\left(\frac{1}{2}\theta\right)$ →

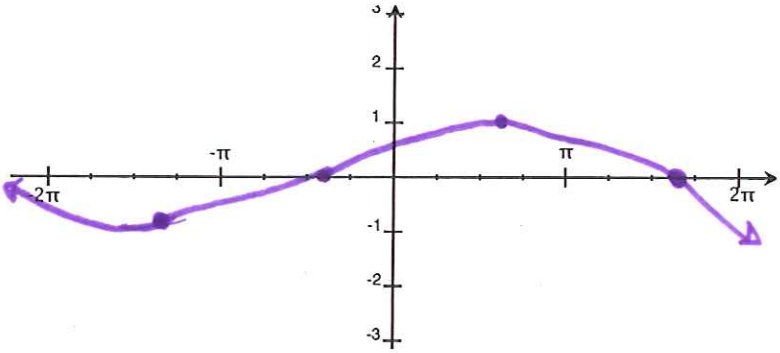
Stretch by 2



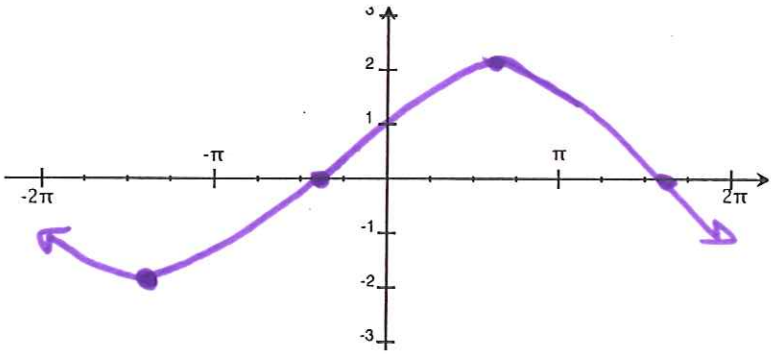
*

$\sin\left(\frac{1}{2}(\theta+\frac{\pi}{3})\right)$ →

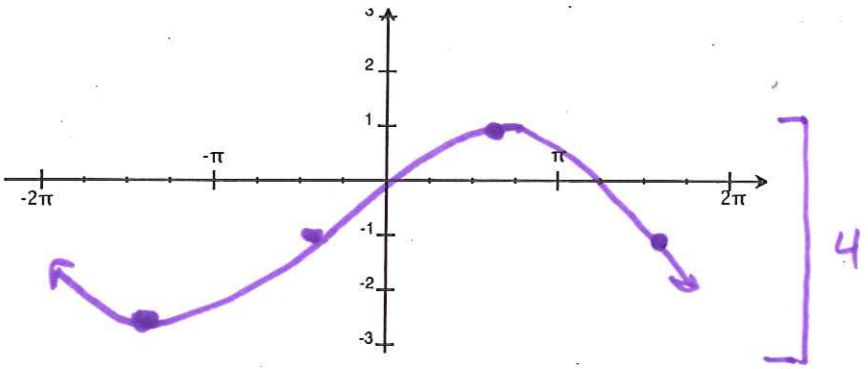
left π/3



$2\sin\left(\frac{1}{2}(\theta+\frac{\pi}{3})\right)$ →



$2\sin\left(\frac{1}{2}(\theta+\frac{\pi}{3})\right)-1$ →



What is the amplitude of $2\sin\left(\frac{1}{2}\theta + \frac{\pi}{6}\right) - 1$? = 2

What is the period? $4\pi = \frac{1}{1/2} * 2\pi$

Trig identities to know and love:

Even/odd:

$$\cos(-\theta) = \cos(\theta) \quad \text{(even)} \qquad \sin(-\theta) = -\sin(\theta) \quad \text{(odd)}$$

Pythagorean identity:

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

Angle addition:

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$

$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$$

(in particular $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ *and* $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$*)*

Check

$$1 = \cos(0) = \cos(0+0)$$

$$\stackrel{?}{=} \cos(0)\cos(0) - \sin(0)\sin(0)$$

$$= 1 * 1 - 0 * 0 = 1 \quad \checkmark$$

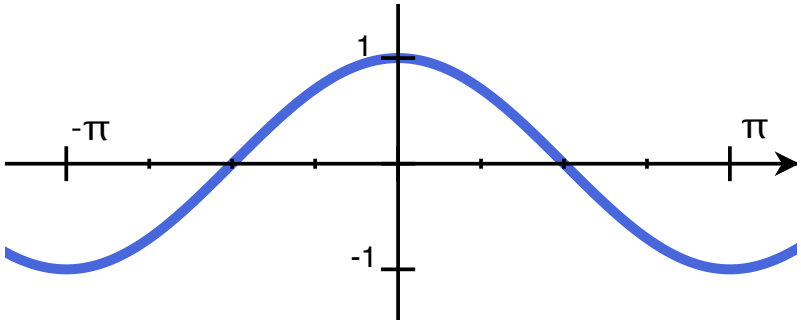
$$0 = \sin(0) = \sin(0+0)$$

$$= \sin(0)\cos(0) + \sin(0)\cos(0)$$

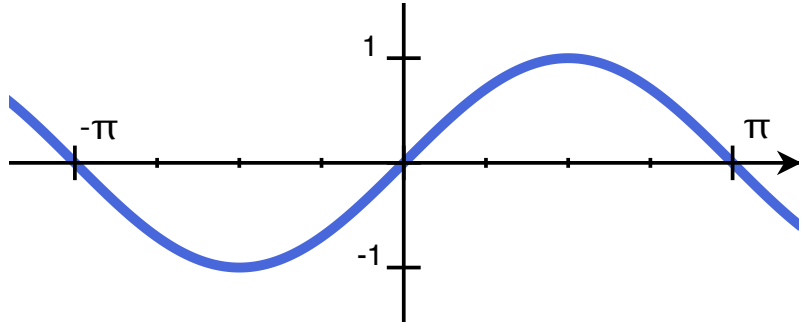
$$= 0 * 1 + 0 * 1 = 0 \quad \checkmark$$

Other trig functions

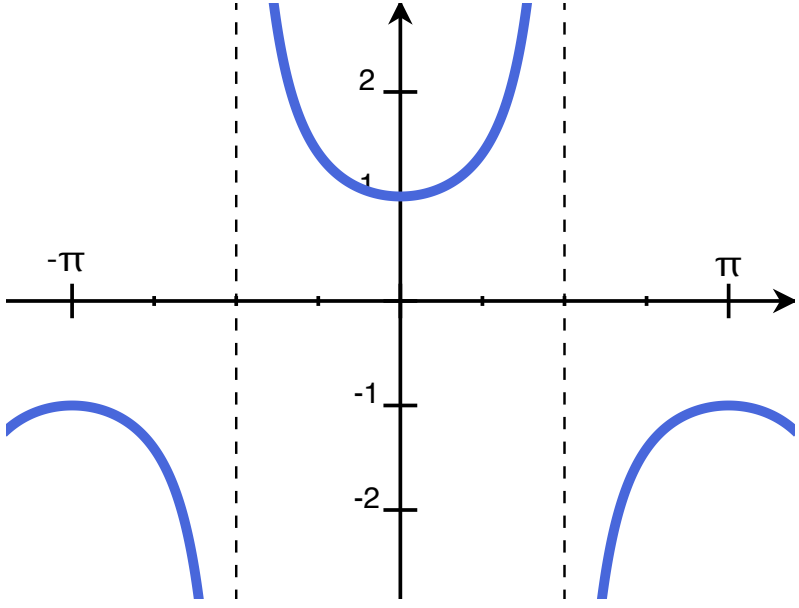
$y = \cos(\theta)$



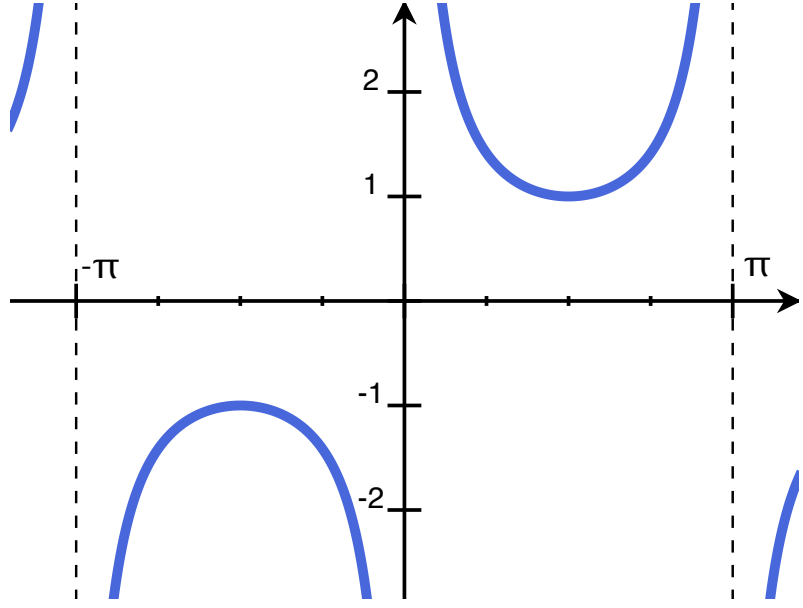
$y = \sin(\theta)$



$\sec(\theta) = 1/\cos(\theta)$

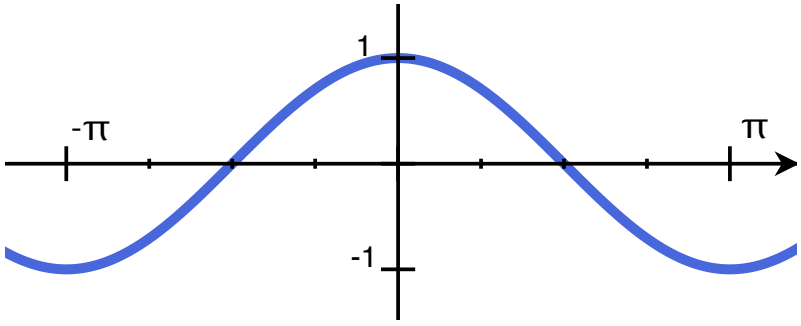


$\csc(\theta) = 1/\sin(\theta)$

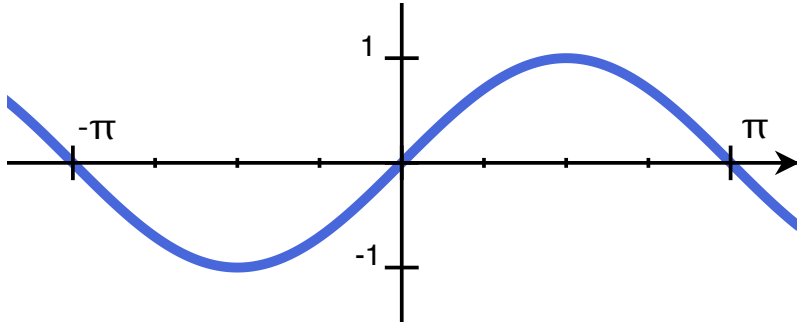


Other trig functions

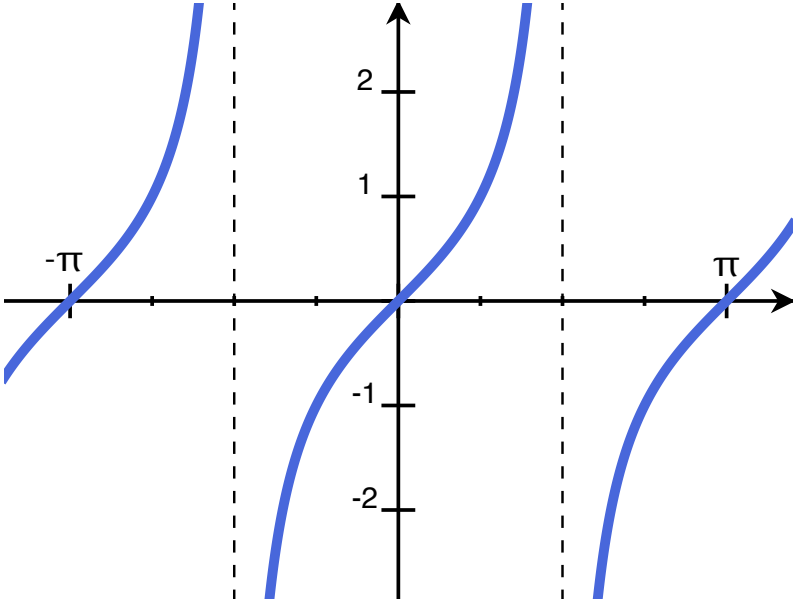
$y = \cos(\theta)$



$y = \sin(\theta)$



$\tan(\theta) = \sin(\theta)/\cos(\theta)$



$\cot(\theta) = \cos(\theta)/\sin(\theta)$