The Parallel Postulate
and an outline of the course

Once upon a time (about 300 BC, specifically), a Greek named Euclid wrote a book called the *Elements*, in which he laid out a system of geometry and proofs of theorems in his system. He had five *postulates*, statements which he presented without proof as givens (the modern term would be *axioms*). These five postulates have hundreds and hundreds of consequences; all of plane geometry stems from them. Here they are:

Let the following be postulated
1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any center and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

These may be modernized a little, and the fifth replaced by a logically equivalent statement, as follows:

1. Any two points may be joined by a straight line.
2. Any finite straight line segment may be extended indefinitely to longer straight line segments.
3. Given any finite line segment, one may draw a circle with that segment as radius and one of its endpoints as center.
4. All right angles are congruent.
5. Given a straight line and a point not on the line, there is exactly one line through the given point parallel to the given line.

The fifth is called the *Parallel Postulate* and was the object of much study in geometry from as early as 100 years after the *Elements*. Many geometers thought only the first four postulates were necessary, and the fifth could be proved from them. None were successful, however.

We should clarify that by "parallel" lines we mean "nonintersecting no matter how far you extend them". For our examples we will also be using "straight line" to mean "shortest distance between two points"; this will not always be a line *per se*, as we will explain in the examples.

Now on to *why* no one was successful in proving the Parallel Postulate from the first four: it does not follow from them. It was in the early nineteenth century, more than two thousand years after Euclid, that Gauss, Bolyai, and Lobachevski all came independently to that conclusion.

How does one show that there is no proof of the fifth postulate from the first four? By showing there are models of geometry in which the first four

postulates hold, but the fifth does not. This may be the reason it took so long for the realization that it did not hold; it is a counterintuitive notion if you are thinking of "geometry" as plane geometry. However, there are numerous examples in the world around us of non-plane geometry. Think of the globe, for instance. What is a "straight line" on a globe? The shortest distance between two points, as any student of airplane routes will know, is a segment of the great circle joining those two points. A great circle is a circle on the surface of a sphere that shares its center with the sphere, or in other words one that has maximal circumference. Therefore "straight lines" on spheres lie on great circles. The first four postulates all hold on the sphere, but any two great circles intersect, so the Parallel Postulate is false.

It is also possible to construct geometries in which the first four postulates hold but there are multiple lines through a given point parallel to a given line. The canonical example here is hyperbolic space, which we won't go into.

There is an important distinction here that we have thus far swept under the rug. Why does constructing geometries in which the Parallel Postulate does or does not hold have any relationship to whether it follows via logical deduction from the first four postulates? This brings us to the subject of this course. We will begin by talking about formalization and what a deduction is; how one manipulates logical formulas and draws conclusions. In this we will distinguish between *syntax* and *semantics*. A syntactic implication is one of symbol manipulation, a proof; it is independent of the "truth" of the premise. Semantic implication occurs when the truth of the premise always indicates the truth of another logical statement. One could think of it as an observation. The Soundness Theorem states that whenever there is a syntactic implication, there is also a semantic one. The Completeness Theorem states the converse, so the two together give us the result that constructing geometric systems in which the first four of Euclid's postulates hold but the fifth does not implies that there is no proof of the fifth postulate from the first four.

Soundness holds in any reasonable axiom system (if you can prove false statements from true ones, the system is useless), but Completeness holds only of systems of a certain simplicity. Once the system gets complicated enough (in particular to be able to make self-referential statements), it may be *incomplete*: in possession of true but unprovable statements. We will discuss incompleteness at the end of the course, in as much depth as we have time for.