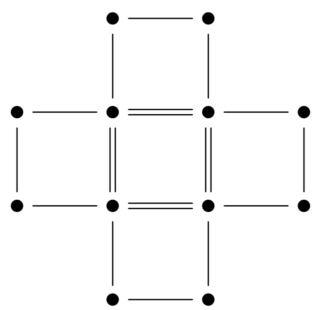
Math 38
Spring 2016
Final Exam Sample Solutions

(1.) For $m, n \geq 3$ we define $G_{n,m}$ to be the graph obtained by starting with a copy of C_n and adding, for each edge e, a copy of C_m containing e and otherwise disjoint from the rest of the graph.

(a.) For $G = G_{n,m}$, find $n(G)$, $e(G)$, $\kappa(G)$, $\kappa'(G)$, $\max\{\lambda'(x,y) \mid (x,y) \in V(G)\}$, $\max\{\lambda(x,y) \mid (x,y) \in V(G), xy \notin E(G)\}$, $\alpha(G)$, and $\beta(G)$.

Solution: First we note that G can be decomposed into two cycles, the initial n-cycle H, and the cycle K consisting of all the remaining edges of G, which also contains all vertices of G. In the picture of $G_{4,4}$, the edges of H are shown with double lines.



H contains n-many vertices, and each of the n-many m-cycles contains $(m-2)$-many vertices that are not also in H, so

$$n(G) = n + n(m-2) = n(m-1).$$

Each of the n-many m-cycles contains m edges, none of which belong to any other m-cycle, and these are all the edges of G, so

$$e(G) = nm.$$

The degree of vertex v of G is 2 if v is not in H, and 4 if v is in H, which together with a theorem in the text gives $\kappa(G) \leq \kappa'(G) \leq \delta(G) = 2$. To show $2 \leq \kappa(G)$, it suffices to show that G is 2-connected. This is true, since removing any single vertex leaves H and K both connected, and connected to each other. Therefore

$$\kappa(G) = \kappa'(G) = 2.$$

If x is a vertex of G not in H, then x is incident to only two edges of G, so if y is any other vertex of G, there can be at most two edge-disjoint paths between x and y. This shows $\lambda(x,y) \leq \lambda'(x,y) \leq 2$. If x and y are both vertices of H, then the maximum size of a pairwise edge-disjoint collection of x,y-paths is 4: It cannot be more than 4, since $d(x) = 4$.

To see it is at least 4, note that x and y are in both cycles H and K (which share no edges), and there are two x, y-paths in each cycle (clockwise and counterclockwise). Therefore,

$$\max\{\lambda'(x, y) \mid (x, y) \in V(G)\} = 4.$$

If x and y are non-adjacent vertices of H, the maximum size of a pairwise internally-disjoint collection of x, y-paths is 2: It cannot be more than 2, since if v and w are the neighbors of x in H, $\{v, w\}$ is an x, y-cut. It is at least 2, since the x, y-paths going in opposite directions around H are internally disjoint. Therefore,

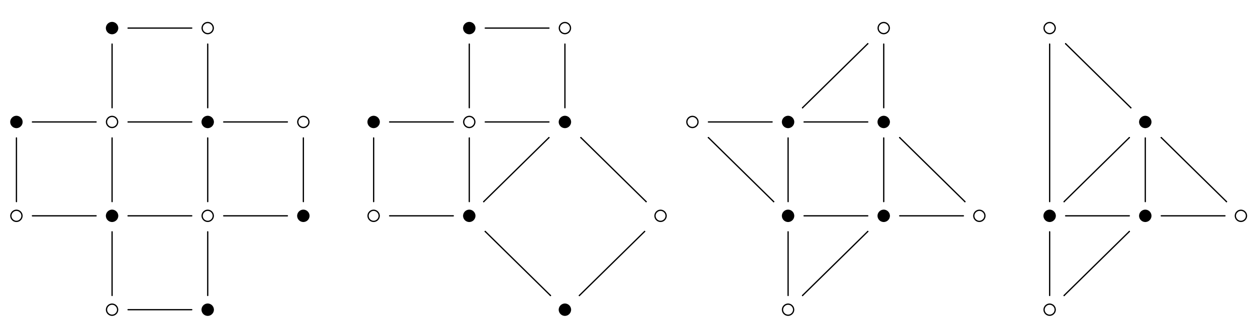
$$\max\{\lambda(x, y) \mid (x, y) \in V(G), xy \notin E(G)\} = 2.$$

A vertex cover of G must include a vertex cover of the $n(m-1)$-cycle K, so must be at least as large as a minimum vertex cover of K, which has size $\left\lceil \frac{n(m-1)}{2} \right\rceil$. We will show that a minimum vertex cover of K can be chosen so that it also covers the edges of H, so it is also a minimum vertex cover of G.

To produce a minimum vertex cover X of K, we begin with any vertex, and proceed around K, including every other vertex in X. Let the vertices of H be $v_1, v_2, \dots v_n$, in order around the cycle; these are also vertices of K. Begin by including vertex v_1 in X, and proceed around K; after $2(m-1)$-many steps, we will reach v_3, and include v_3 in X. (We may or may not include v_2, depending on the parity of m.) Since every odd-indexed vertex of H is included in X, every edge of H is covered by X, so X is a minimum vertex cover of G, of size $\left\lceil \frac{n(m-1)}{2} \right\rceil$. Therefore

$$\beta(G) = \left\lceil \frac{n(m-1)}{2} \right\rceil.$$

Note, this is not the same as $n\left\lceil \frac{(m-1)}{2} \right\rceil$. For $n = m = 4$, we have $n\left\lceil \frac{(m-1)}{2} \right\rceil = 8$ and $\left\lceil \frac{n(m-1)}{2} \right\rceil = 6$; the correct answer is 6. In the pictures, illustrating all cases of the parities of m and n, the solid vertices form minimum vertex covers, beginning with the lower left vertex of H and alternating clockwise around K.



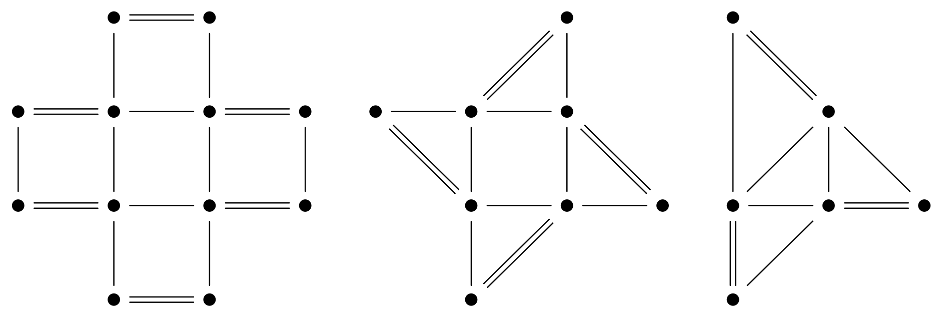
Finally, we have a theorem that $\alpha(G) + \beta(G) = n(G)$. Therefore

$$\alpha(G) = n(G) - \beta(G) = n(m-1) - \left\lceil \frac{n(m-1)}{2} \right\rceil = \left\lfloor \frac{n(m-1)}{2} \right\rfloor.$$

(b.) For what values of m and n does $G_{n,m}$ have a perfect matching?

Solution: If $n(m-1) = n(G_{n,m})$ is odd, then $G_{n,m}$ has no perfect matching, since a graph with a perfect matching must have even order.

If $n(m-1) = n(G_{n,m})$ is even, then $G_{n,m}$ has a perfect matching. One way to get a perfect matching is to take every other edge of the $n(m-1)$-cycle K, which includes every vertex of $G_{n,m}$. In the pictures, the doubled edges form perfect matchings.



(2.) Let G be a bipartite graph with partite sets X and Y. Suppose that for all nonempty $S \subseteq X$, we have $|S| < |N(S)|$. Show that every edge of G belongs to a matching of G that saturates X.

Solution: Let xy be any edge of G, with $x \in X$ and $y \in Y$. The graph $H = G - \{x, y\}$ is bipartite with partite sets $X' = X - \{x\}$ and $Y' = Y - \{y\}$. For any $S \subseteq X'$ the set of neighbors of elements of S in H is $N_H(S) = N_G(S) - \{y\}$. Since $|N_G(S)| > |S|$, we have $|N_H(S)| \geq |N_G(S)| - 1 \geq |S|$.

By Hall's Theorem, then, H has a matching M that saturates $X' = X - \{x\}$. Since all edges of M are also in G, and none is incident to either x or y, we have that $M \cup \{xy\}$ is a matching in G that saturates X and contains xy. Since xy was an arbitrary edge, this proves the proposition.

(3.) Let G be any simple connected graph with at least two vertices. We form G^* as follows:

$$X = \{v_x \mid v \in V(G)\} \qquad Y = \{v_y \mid v \in V(G)\} \qquad V(G^*) = X \cup Y$$

$$E(G^*) = \{v_x w_x \mid vw \in E(G)\} \ \cup \ \{v_y w_y \mid vw \in E(G)\} \ \cup \ \{v_x v_y \mid v \in V(G)\}$$

If $\chi(G) = k$, what can we conclude about $\chi(G^*)$?

Solution: $\chi(G^*) = k$.

Proof: First, since G contains an edge, we know $k \geq 2$.

To show $\chi(G) \leq \chi(G^*)$, let c^* be any proper coloring of G^* in $\chi(G^*)$-many colors. Then $c(v) = c^*(v_x)$ is a proper coloring of G in $\chi(G^*)$-many colors. This is because

$$v \leftrightarrow w \implies v_x \leftrightarrow w_x \implies c^*(v_x) \neq c^*(w_x) \implies c(v) \neq c(w).$$

To show $\chi(G^*) \leq \chi(G) = k$, let c be any vertex coloring of G in k-many colors $1, 2, \ldots, k$. Color G^* with k-many colors by setting

$$c^*(v_x) = c(v);$$

$$c^*(v_y) = \begin{cases} c(v) + 1 & \text{if } c(v) < k; \\ 1 & \text{if } c(v) = k. \end{cases}$$

To see this is a proper coloring: If $v_x w_x \in E(G^*)$, then $vw \in E(G)$, so $c(v) \neq c(w)$, so

$$c^*(v_x) = c(v) \neq c(w) = c^*(w_x).$$

If $v_y w_y \in E(G^*)$, then $vw \in E(G)$, so $c(v) \neq c(w)$, so mod k we have

$$c^*(v_x) \equiv c(v) + 1 \not\equiv c(w) + 1 \equiv c^*(w_x).$$

The only other edges of G^* are of the form $v_x v_y$, and, mod k, we have

$$c^* v_y \equiv c(v) + 1 \not\equiv c(v) = c^*(v_x).$$

Note: Another way to do this problem is to show that G^* is isomorphic to $G \,\Box\, P_2$, and then use Proposition 5.1.11 from the textbook.

(4.) Prove or disprove: G is planar iff every block of G is planar.

Solution: This is true.

For one direction, suppose G is planar. A plane embedding of G is also a plane embedding of every subgraph of G, so every subgraph of G, including every block of G, is planar.

Show the other direction by induction on the number of blocks of G.

For the base case, if G has only one block B, then $G = B$, so if B is planar then G is planar.

For the inductive case, suppose G has k blocks, all planar, and that every graph with fewer than k blocks, all planar, is itself planar.

If G is not connected, then each connected component of G has fewer than k blocks, all of which are planar (since they are also blocks of G), so by inductive hypothesis each connected component is planar. Therefore, G is planar.

If G is connected, then G has some cut vertex v. We can write $G = H \cup K$, where the components of $G - v$ are $H - v$ and $K - v$, and the only vertex H and K have in common is v. Each of H and K has fewer than k blocks, all of which are planar (since they are also blocks of G), so by inductive hypothesis H and K are both planar. If we can find plane embeddings of H and K such that v is on the boundary of the unbounded face in each, then we can find a plane embedding of G: Embed H and K as described, connect the two copies of v with some curve C, then contract C to a point. (Another way to say this: Eliminate the copy of v in the plane copy K, and extend the incident edges along C to reach the copy of v in the plane copy of H.)

So we must prove the following: If H is a planar graph, and v is a vertex of H, then there is a plane copy of H in which the copy of v is on the boundary of the unbounded face.

Proof: Using the method of Remark 6.1.27 in the textbook, embed H in the sphere, and puncture the sphere inside a face F having vertex v on its boundary. This yields a planar embedding in which F is the unbounded face, and v is on its boundary.

(5.) Prove that a tree T has a perfect matching iff T satisfies $(*)$:

(*) For every vertex v, the graph $T - v$ has exactly one component of odd order.

Solution:

For one direction, suppose T has a perfect matching M, let v be any vertex of T, let $vw \in M$, and let the components of $T - v$ be $T_1, \ldots, T_k$, where $w \in T_k$.

If x is any vertex of T_i for $i < k$, and xy is the edge of M incident to x, then the edge xy is an edge of $T - v$, and therefore an edge of T_i. This shows the edges of M in T_i form a perfect matching in T_i, so T_i has even order. Since T has even order, $T - v$ has odd order; since all components of $T - v$ except T_k have even order, T_k must have odd order.

For the other direction, show that if T satisfies $(*)$, then T has a perfect matching. We will do this by induction on the order of T.

If T has only one vertex, then $T - v$ has no vertices, so T does not satisfy $(*)$. If T has two vertices, then T is a single edge, so T has a perfect matching.

Suppose T has n vertices where $n > 2$, and suppose the proposition holds for trees of order less than n. Further suppose that T satisfies $(*)$. We will show that there is an edge vw of T such that $S = T - \{v, w\}$ is a tree satisfying $(*)$. Then, by inductive hypothesis, S has a perfect matching M, and so T has a perfect matching $M \cup \{vw\}$.

Let T' be the tree of internal (non-leaf) vertices of T, let v be a leaf of T', and let u be the neighbor of v in T'. Since v is not a leaf of T, there must be some neighbor w of v that is not in T'; that is, w is a leaf of T. Furthermore, v cannot have two neighbors that are leaves of T; if it did, both would be isolated vertices in $T - v$, and therefore $T - v$ would have at least two components of odd order, contradicting $(*)$. Therefore, v has exactly two neighbors u and w in T, and v is the only neighbor of w in T. We will show $S = T - \{v, w\}$ satisfies $(*)$.

If P is any path in T, the leaf w cannot be an internal vertex of P, and if v is an internal vertex of P, then w must be a endpoint of T. Hence, if x and y are vertices of T other than v and w, the x, y-path in T is also in S; this means that if z is some other vertex of S, the vertices x and y are in the same component of $S - z$ iff they are in the same component of $T - z$ (iff z is not on the x, y-path in T). That is, the components of $T - z$ are exactly the components of $S - z$, with one exception: if $z \neq u$, the component of $T - z$ containing u has the vertices of the component of $S - z$ containing u plus two additional vertices, v and w; if $z = u$, there is an additional component of $T - z$ containing only the vertices v and w. In either case $T - z$ and $S - z$ have the same number of components of odd parity. Therefore, since T satisfies $(*)$, it follows that S also satisfies $(*)$.

Note: Another way to do this: Define an edge e of T to be odd if both components of $T - e$ have odd order. Show that if $T - v$ has exactly one component of odd order, then v

is incident to exactly one odd edge. Conclude that if for every vertex v the graph $T - v$ has exactly one component of odd order, then the odd edges form a perfect matching.

(6.) (a.) Count the number of isomorphism classes of strongly connected digraphs with C_n as underlying graph.

Solution: There is exactly one.

If $n = 1$, then C_n has a single edge which is a loop from v to v, which can only be oriented from v to v.

If $n = 2$, then C_n has a double edge consisting of edges e and f between v and w. There are only two ways to orient the edges of C_2 to get a strongly connected digraph, orienting one edge from v to w and the other from w to v. These digraphs are isomorphic via the automorphism of C_2 that fixes v and w and switches e and f.

Let G be a copy of C_n with vertices $v_1, \ldots, v_n$, arranged counterclockwise around a circle in the plane. There are only two ways to orient the edges of G to get a strongly connected digraph; since each vertex must have an in edge and an out edge, either all edges are oriented clockwise or all are oriented counterclockwise. These digraphs are isomorphic via the automorphism of G that sends $v_1, \ldots, v_n$ to $v_n \ldots, v_1$ in that order.

(b.) Count the number of isomorphism classes of digraphs with P_{2n} as underlying graph. (P_{2n} has $2n$-many vertices.)

Solution: There are 4^{n-1}-many.

Let G be a copy of P_{2n}. G has $(2n-1)$-many edges, each of which can be oriented in either direction, so there are 2^{2n-1}-many digraphs with G as underlying graph. There are two automorphisms of G, the identity automorphism and the automorphism f that switches the endpoints of the path; each digraph H is isomorphic via f to a digraph which we may call $f(H)$. Furthermore, H and $f(H)$ are not the same digraph, since the middle edge is oriented in different directions in H and $f(H)$. Therefore, the 2^{2n-1}-many digraphs with G as underlying graph are partitioned into isomorphism classes each of which has size 2, so the number of isomorphism classes is $\dfrac{2^{2n-1}}{2} = 2^{2n-1} = 4^{n-1}$.

Note: The situation is different in P_{2n+1}, which does not have a middle edge; in this case, some isomorphism classes have size 1 and some have size 2. For example, here are the four digraphs with underlying graph a given copy of P_3, which represent three isomorphism classes; the first two are isomorphic.

(7.) Suppose that G and H are simple Eulerian graphs each of which has at least one edge. Find a condition on $G \cap H$ that will guarantee $G \cup H$ is Eulerian.

Find a condition on $G \cap H$ that will guarantee $G \cup H$ is not Eulerian.

The best answer to this problem would be a proposition of the form "$G \cup H$ is Eulerian iff $G \cap H$ has condition X." If you can't get this, get as close as possible.

Solution: We know $G \cup H$ is Eulerian iff $G \cup H$ is even and has at most one nontrivial component.

For any vertex v of $G \cup H$, setting $d_K(v) = 0$ if v is not a vertex of K, we have

$$d_{G \cup H}(v) = d_G(v) + d_H(v) - d_{G \cap H}(v).$$

Since G and H are even, the right hand side of this equation is even iff $d_{G \cap H}(v)$ is even; it follows that $G \cup H$ is even iff $G \cap H$ is even.

Since G and H each have nontrivial components, $G \cup H$ will have only one nontrivial component iff the nontrivial components of G and H are connected in $G \cup H$. This will happen iff they have at least one vertex in common; that is, iff $G \cap H$ contains some vertex v that is non-isolated in both G and H.

Whether you can actually get a proposition of the form "$G \cup H$ is Eulerian iff $G \cap H$ has condition X" depends on whether you allow your condition on $G \cap H$ to mention G and H.

If you do, you can say $G \cup H$ is Eulerian iff $G \cap H$ is even and contains some vertex v that is non-isolated in both G and H.

If not, you can still say: If $G \cap H$ is not even or is empty, then $G \cup H$ is not Eulerian. If $G \cap H$ is even and contains at least one edge, then $G \cup H$ is Eulerian. (If $G \cap H$ consists of isolated vertices, $G \cup H$ may or may not be Eulerian, depending on whether one of the vertices of $G \cap H$ is non-isolated in both G and H.)