Midterm Exam: Due Friday, May 22 at 1:35pm (at end of class)

- You are free to use your own book and class solutions that I have posted online. You are also permitted to use your own notes and homework.
- You may not communicate with other students or any other people about the problems or any of the course material. You also may not seek assistance online or from other books.
- Organize your solutions and write them neatly!
- Take time to explain your work and justify your solutions. *Constructions and computations should be explicit and must be accompanied by arguments that they have the properties you claim they do.*

Problem 1: (8 points) Let G be a bipartite graph with partite sets X and Y. Suppose that $|S| < |N(S)|$ for all nonempty $S \subseteq X$. Show that every edge of G belongs to a matching of G which saturates X.

Problem 2:
(a) (7 points) Suppose that G is a simple graph on $2n$ vertices with $\delta(G) \geq n$. Show that G has a perfect matching.
(b) (5 points) For each $n \geq 2$, construct a simple graph on $2n$ vertices with $\delta(G) = n - 1$ which has no perfect matching.

Problem 3: Let G be the simple graph with vertex set $[210] = \{1, 2, \ldots, 210\}$ and where two vertices $a, b \in [210]$ with $a < b$ are adjacent if $\frac{b}{a}$ is a prime number (and thus also an integer).
(a) (4 points) Compute $\kappa(G)$.
(b) (4 points) Compute $\kappa(1, 210)$.

Problem 4:
(a) (4 points) Show that if $\kappa'(G) = \ell \geq 2$, then the deletion of ℓ edges from G results in a graph with at most 2 components.
(b) (4 points) In contrast, show that for every $\ell \geq 2$, there exists a graph G with $\kappa(G) = \ell$ which has a set of ℓ vertices whose deletion results in at least 3 components.

Problem 5: (5 points) Suppose that $1 \leq k \leq \ell$. Show that there exists a graph G with vertex x such that $\kappa(G) = k$ and $\kappa(G - x) = \ell$.

Problem 6: (9 points) Let G be a simple graph. Recall that the complement of G, denoted $\overline{G}$, is the simple graph with the same vertex set as G and such that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. Show that $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$ for every simple graph G. (Hint: Induction is handy for the latter inequality).