15. (A voting game) Three voters I, II and III are electing one of four candidates. Their profile is

I	II	III
B	D	A
D	A	B
C	C	C
A	B	D

The voting mechanism is by successive vetoes. Specifically, first Voter I vetoes one of the candidates, then voter II vetoes another, and finally III decides the winner by vetoing one of the remaining two candidates. This defines a 3-person game with possible outcomes A, B, C or D.

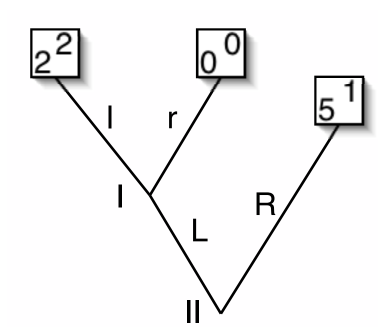
(a) Draw a game tree for this game.

(b) Find a Nash equilibrium using Zermelo's algorithm. Also find at least one Nash equilibrium that is not subgame perfect. Which candidate wins?

(c) Show that, for the profile above, the order in which the voters cast their vetoes does not effect the outcome of the election. (This can be done without drawing any of the six possible game trees.)

(d) Give as simple a profile as possible to show that the winner of an election by successive vetoes may depend on the order in which the voters cast their vetoes.

16. Let (s,t) and (s',t') be two different saddle points of a strictly competitive game with outcomes $u_1 \prec_{\mathrm{I}} u_2 \prec_{\mathrm{I}} \ldots \prec_{\mathrm{I}} u_k$. Prove that (s',t) and (s,t') are also saddle points. How are the outcomes of the game related at the four saddle points?



17. Consider the game

(a) Is it strictly competitive? Why?

(b) What is the strategic form of the game?

(c) Find all Nash equilibria. Of these, which are subgame-perfect?

18. Find the maximin and the minimax values of the matrix $\begin{pmatrix} -3 & -3 & 2 \\ -1 & -3 & -2 \\ 3 & -1 & -2 \\ 2 & 2 & -3 \end{pmatrix}$.

Does it have a saddle point (if only pure strategies are used)? Why?

19. Show that a saddle point (when only pure strategies are used) of a zero sum game remains a saddle point when mixed strategies are used.

20. Find the value of the zero-sum game with payoff matrix $\begin{pmatrix} 1 & -1 & 3 & 2 \\ 3 & 5 & -3 & 3 \end{pmatrix}$. What is an optimal strategy for each of the two players?

21. Here is a simple card game. Player I holds two Kings and an Ace. He sets one card aside, and then announces either KK or KA to let player II know which hand he now has. KA is considered a better hand than KK. Player II may either accept or challenge I's call. If he challenges and catches I lying, the payoff to I is -4 if I claims to have a better hand than he really does. If it's the lesser hand, the payoff is 4. If player I is not lying, the payoff is -2. If II accepts I's call, then, in each case, the payoff is $-\frac{1}{2}$ of what it is if II challenges, except when II accepts I's truthful call of KK. Then the payoff is 0, rather than 1.

    (a) Draw the game tree of this card game.

    (b) Find the value of the game and an optimal strategy for each of the two players.

22. Redo problem 20 above using the computational method suggested by the proof of von Neumann's theorem.

23. Let
$$M = \begin{pmatrix} \pi_1 & 0 & 0 \\ 0 & \pi_2 & 0 \\ 0 & 0 & \pi_3 \end{pmatrix}$$
where the π_i are all positive, be the payoff matrix of a zero-sum game. What is the value of the game and what are optimal strategies for the two players? Can you generalize to a zero-sum game with $n \times n$ payoff matrix in which all diagonal entries are positive and all off-diagonal entries are 0?

24. Using only Nash's axioms, find the Nash solution to the bargaining problem (S, u^*, v^*) where $(u^*, v^*) = (5/2, 7/4)$ and S is the convex hull of the points $(2, 1)$, $(4, 1)$ and $(2, 4)$. (The transformation $u' = \frac{1}{2}u - 1$, $v' = \frac{1}{3}v - \frac{1}{3}$ may be helpful.)

25. Consider the cooperative bimatrix game
$$\begin{pmatrix} (-1,-1) & (1,1) \\ (2,-2) & (-2,2) \end{pmatrix}$$

    (a) Draw the feasible set and find the security level for each player.

    (b) Find the Nash bargaining solution of the game (S, u^*, v^*) where S is the feasible set, and u^* and v^* are the security levels. (This is the Shapley solution of the bimatrix game.)

    (c) What is the outcome of the threat game corresponding to the bimatrix game if player I uses threat strategy $t_{\mathrm{I}} = (1/2, 1/2)$ and II uses $t_{\mathrm{II}} = (1, 0)$? Same question, but with $t_{\mathrm{II}} = (1/2, 1/2)$ and any threat strategy for I.

    (d) Find the optimal threat strategies t_{I}^* and t_{II}^* for the two players and the solution of the corresponding bargaining problem (S, u^*, v^*), where u^* and v^* are the payoffs when t_{I}^* and t_{II}^* are used. (This solution is the Nash solution to the bimatrix game.)

    (e) Which player has the bargaining advantage and why?