1. A certain population initially contains P_0 individuals.

   (a) If left undisturbed the population doubles in size every T days. What differential equation does the population function $P = P(t)$ satisfy? Find $P(t)$.

   (b) Now, to help with part (c), make the very unrealistic assumption that there is no change in the population except that R individuals are removed from it each day at a uniform rate. What differential equation does the population function $P(t)$ now satisfy, and what is $P(t)$?

   (c) Assume now that the population doubles in size every T days if left undisturbed and that R individuals are removed from it each day. What now is the relevant differential equation and its solution $P(t)$?

   (d) Show that the function $P(t)$ found in part (c) is qualitatively different for different values of the ratio P_0/R. What value of P_0/R is critical with regards to the qualitative nature of the function $P(t)$?

   (e) To what extent can part (d) be easily answered without solving the differential equation of part (c)? Is the critical value of P_0/R found in part (d) an equilibrium point? Stable or unstable?

   (f) Describe an ecosystem which part (c) models fairly well.

2.

   (a) Consider an ecosystem consisting of three interacting populations: fish, seals, and Eskimos. Assume that the Eskimos are entirely dependent on seals for food, that the seals eat only fish, and that the fish have an inexhausible food supply. Formulate a system of differential equations, involving E, F,and S the number of Eskimos, fish, and seals respectively, which models this ecosystem reasonably well.

   (b) Repeat part (a), except assume that Eskimos eat both fish and seals.

3. Suppose that we incorporate logistic growth into one of the two populations in the competitive hunters model. Then the relevant equations are

$$\frac{dx}{dt} = a(M - x)x - bxy \quad \text{and} \quad \frac{dy}{dt} = py - cxy.$$

Find the equilibrium points and null clines of these. Sketch enough trajectories in the phase plane so that the qualitative nature of every solution is apparent. How does the long-term behaviour of the solutions depend on their initial values? Which, if any, of the equilibrium points are stable? (Note that there are at least two alternatives depending on whether $M > p/c$ or not. Investigate both cases).

4. Consider the system of differential equations

$$\frac{dx}{dt} = y - cx(x^2 + y^2) \quad \text{and} \quad \frac{dy}{dt} = -x - cy(x^2 + y^2)$$

where c is any positive constant.

(a) Show that the only equilibrium point is $(0,0)$.

(b) Partially solve the system by first combining the two equations to get

$$x\frac{dx}{dt} + y\frac{dy}{dt} = -c(x^2 + y^2)^2 \quad \text{or} \quad \frac{1}{2}\frac{d}{dt}|\mathbf{v}|^2 = -c|\mathbf{v}|^4$$

where $|\mathbf{v}| = \sqrt{x^2 + y^2}$ is the length of the vector $\mathbf{v} = \langle x, y \rangle$. Then think of $|\mathbf{v}|$ as the unknown function in the last differential equation, and take the derivative of $|\mathbf{v}|^2$ in it to get another differential equation. Solve this for $|\mathbf{v}| = |\mathbf{v}(t)|$. What is $\lim_{t \to \infty} |\mathbf{v}(t)|$? Is $(0,0)$ a stable or unstable equilibrium? Why?

(c) What is the linearization of the system at $(0,0)$?

(d) Find the exact trajectories of the linearized system. Is $(0,0)$ a stable or unstable equilibrium of the linearized system? How do the system and its linearization compare in this regard?

5. Consider the system

$$\frac{dx}{dt} = (1-x)x - 2xy \quad \text{and} \quad \frac{dy}{dt} = (1-y)y - 2xy.$$

It models competitive hunters, or competing populations in general, with a growth constraint on each population. It's identical to the system considered in class except for the constants in the terms containing xy.

(a) Find the equilibrium points and determine which are stable and which are unstable.

(b) Can the populations coexist or will one become extinct in the long run? Does the answer to this depend on the initial conditions? You may need to sketch some trajectories for this, or at least some null clines.

(c) How would you expect the answer to part (b) to depend on the constants in front of the xy's in the system? Does your work confirm your expectations?