Math 35: Real Analysis
Winter 2018

Monday 01/22/18

Lecture 8

Chapter 2 - Sequences

Chapter 2.1 - Convergent sequences

Aim: Give a rigorous definition of convergence for sequences.

Definition 1 A **sequence** (of real numbers)

$$a : \mathbb{N} \to \mathbb{R}, n \mapsto a(n)$$

is a function from the natural numbers to the real numbers. Though it is a function it is usually denoted as a list

$$(a_n)_{n\in\mathbb{N}} \quad \text{or} \quad (a_n)_n \quad \text{or} \quad \{a_n\} \textbf{ (notation from the book)}$$

The numbers $a_1, a_2, a_3, \ldots$ are called the **terms** of the sequence.

Example 2: Find the first five terms of the following sequences an then sketch the sequence a) in a dot-plot.

a) $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$.

b) $\left(\frac{2^n}{n!}\right)_{n\in\mathbb{N}}$.

c) the sequence $(a_n)_{n\in\mathbb{N}}$ defined by

$$a_1 = 1, a_2 = 1 \quad \text{and} \quad a_n = a_{n-1} + a_{n-2} \quad \text{for all} \quad n \geq 3. \quad \textbf{(Fibonacci sequence)}$$

Similar as for functions from $\mathbb{R}$ to $\mathbb{R}$ we have the following definitions for sequences:

Definition 3 (bounded sequences) Let $(a_n)_n$ be a sequence of real numbers then

a) the sequence $(a_n)_n$ is **bounded above** if there is an $M \in \mathbb{R}$, such that

$$a_n \leq M \quad \text{for all} \quad n \in \mathbb{N}\,.$$

In this case M is called an **upper bound** of $(a_n)_n$.

b) the sequence $(a_n)_n$ is **bounded below** if there is an $m \in \mathbb{R}$, such that

$$m \leq a_n \quad \text{for all} \quad n \in \mathbb{N}\,.$$

In this case m is called a **lower bound** of $(a_n)_n$.

c) the sequence $(a_n)_n$ is **bounded** if there is an $\tilde{M} \in \mathbb{R}$, such that

$$|a_n| \leq \tilde{M} \quad \text{for all} \quad n \in \mathbb{N}\,.$$

In this case $\tilde{M}$ is called a **bound** of $(a_n)_n$.

Note: This means a sequence $(a_n)_{n\in\mathbb{N}}$ is bounded from above \ bounded from below \ bounded if and only if the set $S = \{a_n, n \in \mathbb{N}\}$ is bounded from above \ bounded from below \ bounded.

Exercise: Categorize the three sequences from **Example 2**.

Definition 4 (monotone sequences) Let $(a_n)_n$ be a sequence of real numbers then

a) the sequence $(a_n)_n$ is **increasing** if

$$a_n \leq a_{n+1} \quad \text{for all} \quad n \in \mathbb{N}$$

and **strictly increasing** if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.

b) the sequence $(a_n)_n$ is **decreasing** if

$$a_n \geq a_{n+1} \quad \text{for all} \quad n \in \mathbb{N}$$

and **strictly decreasing** if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.

c) the sequence $(a_n)_n$ is **monotone** if it is increasing or decreasing and **strictly monotone** if it is strictly increasing or strictly decreasing.

Exercise: Which of the sequences from **Example 2** are monotone? Prove that the sequence 2b) is strictly decreasing for $n \geq 2$ and that the sequence in 2c) is strictly increasing for $n \geq 2$.

Definition 5 A sequence $(a_n)_n$ **converges to a number** a if for all $\epsilon > 0$ there is an $N(\epsilon) = N \in \mathbb{N}$ such that

$$|a - a_n| < \epsilon \text{ for all } n \geq N(\epsilon)$$

The number a is called the **limit** of the sequence and we write

$$\lim_{n\to\infty} a_n = a.$$

The sequence $(a_n)_n$ is called **convergent** if there is a number a, such that the sequence converges to a. It is called **divergent** if it does not converge.

Note: This means that for a given ϵ all values of $(a_n)_n$ must satisfy

$$a_n \in (a - \epsilon, a + \epsilon) \textbf{ for all } n \geq N(\epsilon).$$

Example: Consider the sequence $\left(\frac{(-1)^n}{n}\right)_{n\in\mathbb{N}}$ with limit 0.
Find $N(\frac{1}{5})$ and $N(\frac{1}{2})$. Is it sufficient to make a single calculation?
Then show that this sequence is convergent. Finally, draw a picture explaining **Definition 3** in this case.

Math 35: Real Analysis
Winter 2018

Monday 01/22/18

Theorem 6 If a sequence $(a_n)_n$ is convergent, then the limit is unique.

proof: By contradiction: Suppose that $(a_n)_n$ has two limits, a and b, where $a < b$ i.e $b - a > 0$. Take $\epsilon = \frac{b-a}{4}$.

Theorem 7 A convergent sequence $(a_n)_n$ is bounded.

proof: Idea: The first n values are bounded and the remainder lie in a neighborhood of the limit. Take $\epsilon = 1$.

Note: The statement $(a_n)_n$ is bounded $\Rightarrow$ $(a_n)_n$ converges is not true.
Couterexample:

In a similar fashion as in **Definition 3** we define when the limit of a sequence is infinity.

Definition 8 Let $(a_n)_n$ be a sequence. We say that

1.) The limit of the sequence $(a_n)_n$ is **infinity** or $\lim_{n\to\infty} a_n = \infty$ if for all $M > 0$ there is $N = N(M)$, such that
$$a_n > M \quad \text{for all} \quad n \geq N(M).$$

2.) The limit of the sequence $(a_n)_n$ is **minus infinity** or $\lim_{n\to\infty} a_n = -\infty$ if for all $M < 0$ there is $N = N(M)$, such that
$$a_n < M \quad \text{for all} \quad n \geq N(M).$$

Theorem 9 Let $(a_n)_n$ and $(b_n)_n$ be two convergent sequences, such that
$$\lim_{n\to\infty} a_n = a \quad \text{and} \quad \lim_{n\to\infty} b_n = b.$$

Then we have

a) For all $c \in \mathbb{R}$ we have: $\lim_{n\to\infty} c \cdot a_n = c \cdot a$.

b) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = a + b$.

c) $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = a \cdot b$.

d) If $b_n \neq 0$ for all n and $b \neq 0$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} \frac{1}{b_n} = \frac{a}{b}$.

proof:

Lecture 9

Example: Use the fact that $\lim_{n\to\infty} \frac{1}{n} = 0$ and **Theorem 9** to show that

$$\lim_{n\to\infty} a(n) = \lim_{n\to\infty} \frac{3n^2 + 2n - 1}{n^2 + n} = 3.$$

Hint: Divide first the numerator and denominator of $a(n)$ by n^2.

Theorem 10 (Squeeze Theorem) Let $(a_n)_n$ and $(b_n)_n$ be two convergent sequences. Let $(x_n)_n$ be a sequence, such that

$$a_n \leq x_n \leq b_n \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n.$$

Then $(x_n)_n$ converges and $\lim_{n\to\infty} x_n = x$.

proof: For fixed $\epsilon > 0$ we know that

$$|a_n - x| < \epsilon \quad \text{for all } n \geq N_1(\epsilon) \quad \text{and} \quad |b_n - x| < \epsilon \quad \text{for all } n \geq N_2(\epsilon)$$

Especially for $n \geq N = \max\{N_1(\epsilon), N_2(\epsilon)\}$ we have

$$x - \epsilon < a_n \leq x_n \leq b_n < x + \epsilon \quad (*)$$

This means that $|x - x_n| < \epsilon$ for all $n \geq N$. □

Example: Use the Squeeze Theorem to show that $\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$.
Then sketch the sequences involved in a dot-plot.

Math 35: Real Analysis
Winter 2018

Monday 01/22/18

Chapter 2.2 - Montone and Cauchy sequences

Aim: A sequence converges if and only if its elements approach each other. This is a consequence of the completeness of $\mathbb{R}$.

Theorem 1 A monotone sequence converges, if and only if it is bounded.

proof: " $\Rightarrow$ " A convergent sequence is bounded by **2.1.Theorem 7**, hence a monotone sequence is bounded.

" $\Leftarrow$ " We prove the statement for an increasing sequence $(b_n)_n$. So we know that

$$b_n \le b_{n+1} \quad \text{for all} \quad n \in \mathbb{N}\,.$$

Note: The key is to use the Completeness Axiom to show that an increasing sequence $(b_n)_n$ has a supremum β. We showed that $\lim_{n\to\infty} b_n = \beta$.

Math 35: Real Analysis
Winter 2018

Monday 01/22/18

Exercise 2: Consider the sequence $(b_n)_{n\in\mathbb{N}}$ defined by $b_0 = 2$ and $b_{n+1} = 5 - \frac{4}{b_n}$.

a) Find the first four terms of the sequence.

b) Suppose that $\lim_{n\to\infty} b_n = b$, where $b \neq 0$. Using the definition of the sequence can you find all possible limits?

c) Show by induction that all values of the sequence are contained in the interval $[1, 4]$.

d) Show by induction that the sequence is increasing.

e) Use **Theorem 1** and conclude that the limit exists. What is the value ?