Math 35: Real Analysis
Winter 2018

Wednesday 01/17/18

Lecture 6

Last time:

Lemma 4 The following statements are equivalent:

1. If $a, b > 0$ then there is a positive integer $n \in \mathbb{N}$, such that $na > b$.
2. The set $\mathbb{N}$ of positive integers is not bounded above.
3. For each $x \in \mathbb{R}$ there is an integer $n \in \mathbb{Z}$, such that $n \leq x < n + 1$.
4. For each $x \in \mathbb{R}^+$ there is a positive integer $n \in \mathbb{N}$, such that $\frac{1}{n} < x$.

Example: Show that $\inf(S) = 0$ for $S = \{\frac{1}{n}, n \in \mathbb{N}\}$.

Theorem 5 For all $x, y \in \mathbb{R}, x < y$ there is $q \in \mathbb{Q}$ and $r \in \mathbb{R} \setminus \mathbb{Q}$, such that

$$x < q < y \quad \text{and} \quad x < r < y.$$

proof:

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Chapter 1.6 - Countable and uncountable sets

Aim: $\mathbb{Q}$ and $\mathbb{R}$ are infinite, but $\mathbb{R}$ is "more" infinite.

To compare infinte sets, we can not just use counting arguments. We compare them by showing that there is a map that is one-to-one correspondence or bijective.

Definition 1 Let A and B be sets. A function $f : A \to B$ is a map that assigns to each element of a exactly one element $f(a) = b$.

Figure:

Definition 2 A function $f : A \to B$ is called

a) **injective** if for all $b \in B$ there is at most one element $a \in A$, such that $f(a) = b$. This is equivalent to the statement: $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

b) **surjective** if for all $b \in B$ there is an element $a \in A$, such that $f(a) = b$.
We can check for surjectivity by finding an element $a \in A$, such that $f(a) = b$.

c) **bijective** or in **one-to-one correspondence** if for all $b \in B$ there is exactly one element $a \in A$, such that $f(a) = b$.

Figure:

Note 3: A function $f : A \to B$ is bijective if and only if there is a function $g : B \to A$, such that

$$f \circ g(b) = b \text{ for all } b \in B \quad \text{and} \quad g \circ f(a) = a \text{ for all } a \in A.$$

In this case we call $g = f^{-1}$ the **inverse function** of f.

Note 4: If $f : A \to B$ and $g : B \to C$ are bijective functions, then $g \circ f : A \to C$ is also a bijective function.

proofs: see book, Appendix B.2.

Example 5: Find a bijective function

a) $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$.

b) $h : \mathbb{N} \cup \{0\} \to 2\,\mathbb{Z}$.

c) $g : \mathbb{Z} \to 2\,\mathbb{Z}$.

d) $w : \mathbb{N} \to \mathbb{Z}$.

e) between all possible permutations of the numbers $\{1, 2, 3, 4\}$ and a subset of $\mathbb{N}$.

We are now able to compare infinite sets via bijections or one-to-one correspondences. We define:

Definition 6 Let A be an arbitrary set.

a) The set A is **finite**, if there is a bijective map $f : A \to \{1, 2, 3, \ldots, n\}$ for some positive integer n.

b) The set A is **infinite**, if it is not finite.

c) The set A is **countably infinite**, if there is a bijective map $f : A \to \mathbb{N}$.

d) The set A is **countable**, if it is either finite or countably infinite.

e) The set A is **uncountable**, if it is not countable.

Example: Categorize the examples from **Example 5**. What about $\mathbb{Q}$ and $\mathbb{R}$?

Theorem 7 A subset of a countably infinite set is countable.

proof: