Math 35: Real Analysis
Winter 2018

Wednesday 01/10/18

Lecture 4

Organization:
- HW 1 due today outside KH 008 at 2 pm
- Next lecture Tuesday, 01/16/18 from 12:15-1:05 pm

Last time: $S = \{a_1, a_2, a_3, \ldots, a_n\} \subset \mathbb{R}$ ($a_i \geq 0$ for all i) then

$$AM(S) = \frac{a_1 + a_2 + \ldots + a_n}{n} \quad \text{and} \quad GM(S) = (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n}}$$

denotes the **arithmetic** and **geometric mean** of S, respectively.

Aim: Prove that $AM(S) \geq GM(S)$.

We have the following lemma:

Lemma 8: Let $n \geq 2$ be an integer. Suppose that $b_1 \leq b_2 \leq \ldots \leq b_n$ are positive real numbers that are not all equal. We have:

$$\text{If } \; b_1 \cdot b_2 \cdot \ldots \cdot b_n = 1, \; \text{ then } \; b_1 + b_2 + \ldots + b_n > n.$$

proof: By induction:

1.) Induction start ($n = 2$): We know that $b_1 \neq b_2$ and $b_1 \cdot b_2 = 1$. Then

$$0 < \left((b_1)^{\frac{1}{2}} - (b_2)^{\frac{1}{2}}\right)^2 = b_1 - 2(b_1 \cdot b_2)^{\frac{1}{2}} + b_2 = b_1 - 2 + b_2.$$

This implies that $2 < b_1 + b_2$. Hence our statement is true for $n = 2$.

2.) Induction step ($n \to n+1$): We have to show that if the statement is true for n numbers, i.e.:

$$\text{If } \; b_1 \cdot b_2 \cdot \ldots \cdot b_n = 1, \; \text{ then } \; b_1 + b_2 + \ldots + b_n > n \quad \textbf{(Induction hypothesis)} \qquad (*)$$

then it is also true for $n + 1$ numbers.
So we assume that $(*)$ is true and we know that

$$b_1 \cdot b_2 \cdot \ldots \cdot b_n \cdot b_{n+1} = 1.$$

We have to show that $b_1 + b_2 + \ldots + b_n + b_{n+1} > n + 1$.
As the numbers are not all equal we know that $b_1 < 1 < b_{n+1}$, as their product is equal to one. Set $B_1 = b_1 \cdot b_{n+1}$ then

$$B_1 \cdot b_2 \cdot \ldots \cdot b_n = 1.$$

Hence we can use our induction hypothesis (*) and obtain

$$B_1 + b_2 + \ldots + b_n \geq n \quad \text{or} \quad b_1 \cdot b_{n+1} + b_2 + \ldots + b_n \geq n \qquad (**)$$

Now we look at the product

$$A = (b_{n+1} - 1) \cdot (1 - b_1) = b_1 + b_{n+1} - b_1 \cdot b_{n+1} - 1, \qquad (***)$$

where $A > 0$ as $b_1 < 1 < b_{n+1}$.

$$\begin{aligned} b_1 + b_2 + \ldots + b_n + b_{n+1} &= b_1 \ \underline{-b_1 \cdot b_{n+1} + b_1 \cdot b_{n+1}} + b_2 + \ldots + b_n + b_{n+1} \\ &= (b_1 \cdot b_{n+1} + b_2 + \ldots + b_n) + (b_{n+1} + b_1 - b_1 \cdot b_{n+1} - 1) + 1 \\ &\overset{(**),(***)}{\geq} n + A + 1 > n + 1. \end{aligned}$$

This proves our statement. □

Theorem 9 (AM-GM inequality) $S = \{a_1, a_2, a_3, \ldots, a_n\} \subset \mathbb{R}$ ($a_i \geq 0$ for all i) set of non-negative real numbers then

$$GM(S) = (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n}} \leq \frac{a_1 + a_2 + \ldots + a_n}{n} = AM(S).$$

Furthermore equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

proof Clearly equality occurs if all numbers are equal and the inequality is trivial if one of the a_k is equal to zero. So we assume that the a_k are not all equal and set $r = GM(S)$. Then

$$\frac{a_1}{r} \cdot \frac{a_2}{r} \cdot \ldots \cdot \frac{a_n}{r} = \frac{a_1 \cdot a_2 \cdot \ldots \cdot a_n}{r^n} = 1.$$

Hence by the previous lemma we have that

$$\frac{a_1}{r} + \frac{a_2}{r} + \ldots + \frac{a_n}{r} > n \quad \text{hence} \quad \frac{a_1 + a_2 + \ldots + a_n}{n} > r = GM(S).$$

This proves our theorem.

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Example: (Solving a calculus problem without calculus). Among all rectangular boxes with fixed volume V, find the dimensions of the box whose surface area S is minimal and determine S in terms of V.
Solution: For a box with side-lengths x, y and z, we have that

$$V = xyz \quad \text{and} \quad S = 2xy + 2xz + 2yz.$$

We know that by **Theorem 9** with $a_1 = 2xy, a_2 = 2xz, a_3 = 2yz$:

$$(8 \cdot (xy \cdot xz \cdot yz))^{\frac{1}{3}} \leq \frac{S}{3} \quad \text{hence} \quad 6 \cdot V^{\frac{2}{3}} \leq S.$$

Furthermore equality occurs if and only if $xy = xz = yz$ or $x = y = z$. Hence for fixed volume the box with minimal surface area has area $6 \cdot V^{\frac{2}{3}}$. It is the box with equal sides.

Note: This problem can, for example, be solved using Lagrange-multipliers and the solution is very lengthy. With **Theorem 9** the proof reduces to two lines.

Theorem 10 (Cauchy-Schwarz inequality) Let $\mathbf{a} = (a_1, a_2, \ldots a_n) \in \mathbb{R}^n$ and $\mathbf{b} = (b_1, b_2, \ldots b_n) \in \mathbb{R}^n$ be two vectors. Then

$$|\mathbf{a} \bullet \mathbf{b}|^2 \leq \|\mathbf{a}\|^2 \cdot \|\mathbf{b}\|^2, \qquad \text{where}$$

$$\mathbf{a} \bullet \mathbf{b} = \sum_{k=1}^{n} a_k \cdot b_k \qquad \text{and} \qquad \|\mathbf{a}\| = (\mathbf{a} \bullet \mathbf{a})^{\frac{1}{2}} = \left(\sum_{k=1}^{n} a_k^2 \right)^{\frac{1}{2}}.$$

proof By **Theorem 9** we have for all $k \in \{1, 2, \ldots, n\}$:

$$\frac{1}{2} \cdot \left(\frac{a_k^2}{\|\mathbf{a}\|^2} + \frac{b_k^2}{\|\mathbf{b}\|^2} \right) \geq \left(\frac{a_k^2}{\|\mathbf{a}\|^2} \cdot \frac{b_k^2}{\|\mathbf{b}\|^2} \right)^{\frac{1}{2}} = \frac{|a_k|}{\|\mathbf{a}\|} \cdot \frac{|b_k|}{\|\mathbf{b}\|}.$$

Summing up over all k this implies

$$\sum_{k=1}^{n} \frac{|a_k|}{\|\mathbf{a}\|} \cdot \frac{|b_k|}{\|\mathbf{b}\|} \leq \sum_{k=1}^{n} \frac{1}{2} \cdot \left(\frac{a_k^2}{\|\mathbf{a}\|^2} + \frac{b_k^2}{\|\mathbf{b}\|^2} \right) = \frac{1}{2} \left(\underbrace{\sum_{k=1}^{n} \frac{a_k^2}{\|\mathbf{a}\|^2}}_{=1} + \underbrace{\sum_{k=1}^{n} \frac{b_k^2}{\|\mathbf{b}\|^2}}_{=1} \right) = 1.$$

Hence

$$\left| \sum_{k=1}^{n} a_k \cdot b_k \right| \overset{HW2}{\leq} \sum_{k=1}^{n} |a_k| \cdot |b_k| \leq \|\mathbf{a}\| \cdot \|\mathbf{b}\|.$$

The Cauchy-Schwarz inequality follows by squaring both sides, as $f(x) = x^2$ is an increasing function on $\mathbb{R}^+$, i.e. $x \leq y \Rightarrow x^2 \leq y^2$ for all $x, y \in \mathbb{R}^+$.
