Math 35: Real Analysis
Winter 2018

Monday 03/05/18

Lecture 26

Aim: We prove the integration laws. We close this chapter with an outlook on Fourier series.

Theorem 5 (Integration by substitution) Let $g : [a, b] \to [c, d]$ be differentiable on $[a, b]$ and g' continuous on $[a, b]$. Let $f : [c, d] \to \mathbb{R}$ be a continuous function. Then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(t)\, dt.$$

proof Idea: This is a consequence of the chain rule. Let $F : [c, d] \to \mathbb{R}$ be a primitive of f i.e $F' = f$. Then by the **Chain rule of differentiation** we have

$$(F \circ g)'(x) = F((g(x))' = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x).$$

Integrating both sides of the above equation we obtain by **Theorem 4**

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_a^b (F \circ g)'(x)\, dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a)) \overset{Th.4}{=} \int_{g(a)}^{g(b)} f(t)\, dt.$$

This proves our statement.

Examples Using integration by substitution calculate the integrals

$a)\ \int_a^b \frac{g'(x)}{g(x)}\, dx \quad$ (for $g > 0$) $\qquad$ b) $\int_0^b \tan(x)\, dx = \int_0^b \frac{\sin(x)}{\cos(x)}\, dx$ where $b < \frac{\pi}{2}$.

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Theorem 6 (Integration by parts) Let $f, g : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ and f', g' continuous on $[a, b]$. Then

$$\int_a^b f'(x) \cdot g(x)\, dx = f(x) \cdot g(x)\Big|_a^b - \int_a^b f(x) \cdot g'(x)\, dx.$$

proof Idea: This is a consequence of the product rule. By the **product rule of differentiation** we have for $f \cdot g$.

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x).$$

Integrating both sides on $[a, b]$ we obtain by **Theorem 4**

$$f(x) \cdot g(x)\Big|_a^b = \int_a^b (f(x) \cdot g(x))'\, dx = \int_a^b f'(x) \cdot g(x)\, dx + \int_a^b f(x) \cdot g'(x)\, dx.$$

Our theorem then follows from this equation.

Example Calculate $\int_a^b \sin^2(x)\, dx$ with the help of **Theorem 6**.

$$\begin{aligned} \int_a^b \sin^2(x)\, dx &= \int_a^b (-\cos(x))' \cdot \sin(x) = -\cos(x) \cdot \sin(x)\Big|_a^b + \int_a^b \cos(x) \cdot \cos(x)\, dx \\ \int_a^b \cos(x) \cdot \cos(x)\, dx &= \int_a^b 1 - \sin^2(x)\, dx = \int_a^b 1\, dx - \int_a^b \sin^2(x)\, dx = x\Big|_a^b - \int_a^b \sin^2(x)\, dx \text{ hence} \\ 2\int_a^b \sin^2(x)\, dx &= x - \cos(x) \cdot \sin(x)\Big|_a^b \quad \text{or} \quad \int_a^b \sin^2(x)\, dx = \frac{x - \cos(x) \cdot \sin(x)}{2}\Big|_a^b. \end{aligned}$$

Hence a primitive of $\sin^2(x)$ is $\frac{x - \cos(x) \cdot \sin(x)}{2}$.

As an application of the **Theorem 6** we want to show the expression for the **Sawtooth function**:

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \frac{\pi - x}{2} \quad \text{for all} \quad x \in (0, 2\pi). \qquad (1)$$

This is an example of a **Fourier series**. A Fourier series is an expression of a function f as an infinite sum of sine and cosine functions. We first show the following theorem.

Theorem 7 Let $g : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ and g' continuous on $[a,b]$. For $k \in \mathbb{R}$ we set

$$F(k) = \int_a^b g(x)\sin(kx)\,dx. \quad \text{Then} \quad \lim_{|k|\to\infty} F(k) = 0.$$

Figure: If we let sin oscillate faster and faster, then the positive and negative parts annihilate each other when we integrate.

proof Using integration by parts with $f'(x) = \sin(kx)$ we obtain

$$F(k) = \int_a^b g(x)\sin(kx)\,dx = -\frac{\cos(kx)}{k} \cdot g(x)\Big|_a^b + \int_a^b g'(x)\frac{\cos(kx)}{k}\,dx. \qquad (*)$$

As both g and g' are continuous on $[a,b]$ they attain their max and min on $[a,b]$ and are bounded. Hence there is a constant M, such that

$$|g(x)| \leq M \quad \text{and} \quad |g'(x)| \leq M \quad \text{for all} \quad x \in [a,b]$$

As $|\cos(kx)| \leq 1$ we get from (*) the estimate

$$|F(k)| \leq \frac{1}{|k|} \cdot (2M + M \cdot (b-a))$$

Hence $\lim_{|k|\to\infty} |F(k)| = 0$ which implies our statement.

Lemma 8 (A trigonometric formula) If $t \in \mathbb{R}$ is not a multiple of 2π. Then for any $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} \cos(kt) = \frac{\sin\left((n+\frac{1}{2}) \cdot t\right)}{2\sin\left(\frac{t}{2}\right)} - \frac{1}{2}.$$

proof see **Resources**.

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proof of (1) We can now prove that $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \frac{\pi - x}{2}$. (see **Figure 1**)

1.) We first note that $\frac{\sin(kx)}{k}$ is a primitive of $\cos(kx)$, especially, as $\sin(k\pi) = 0$ we have

$$\frac{\sin(kx)}{k} = \int_{\pi}^{x} \cos(kt)\,dt. \quad (**)$$

Together with **Lemma 8** we obtain for $n \in \mathbb{N}$

$$\sum_{k=1}^{n} \frac{\sin(kx)}{k} \overset{(**)}{=} \sum_{k=1}^{n} \int_{\pi}^{x} \cos(kt)\,dt = \int_{\pi}^{x} \sum_{k=1}^{n} \cos(kt)\,dt \overset{\textbf{Lemma 8}}{=} \int_{\pi}^{x} \frac{\sin\left(\left(n+\frac{1}{2}\right)\cdot t\right)}{2\sin\left(\frac{t}{2}\right)} - \frac{1}{2}\,dt$$

$$= \int_{\pi}^{x} \underbrace{\frac{1}{2\sin\left(\frac{t}{2}\right)}}_{=g(x)\,in\,\textbf{Th.7}} \cdot \sin(\underbrace{\left(n+\frac{1}{2}\right)}_{=k\,in\,\textbf{Th.7}} \cdot t)\,dt - \int_{\pi}^{x} \frac{1}{2}\,dt = F_x\left(n+\frac{1}{2}\right) + \frac{\pi - x}{2}.$$

2.) For any $x \in (0, 2\pi)$ we can now apply **Theorem 7** to $F_x(n+\frac{1}{2}) = \int_{\pi}^{x} \frac{1}{2\sin\left(\frac{t}{2}\right)} \cdot \sin\left(\left(n+\frac{1}{2}\right)\cdot t\right)\,dt$ and obtain

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin(kx)}{k} = \underbrace{\lim_{n\to\infty} F_x\left(n+\frac{1}{2}\right)}_{=0\;\;by\,\textbf{Th. 7}} + \lim_{n\to\infty} \frac{\pi - x}{2} = \frac{\pi - x}{2}.$$

This proves our statement.

Corollary 9 For $x = \frac{\pi}{2}$ we obtain the formula $\sum_{k=1}^{\infty} \frac{\sin(k\cdot\frac{\pi}{2})}{k} = \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}}$.

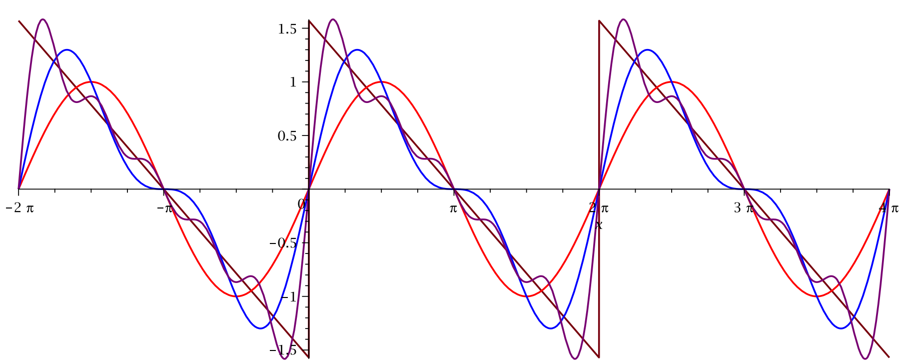

Figure 1: Plot of $\frac{\pi - x}{2}$ (black) and the approximations with sine functions $\sin(x)$ (red), $\sin(x) + \frac{\sin(x)}{2}$ (blue) and $\sum_{k=1}^{5} \frac{\sin(kx)}{k}$ (purple) for $x \in (0, 2\pi)$ and neighboring intervals.