Math 35: Real Analysis
Winter 2018

Friday 03/02/18

Lecture 25

For $g(x) = 1$ for all $x \in [a, b]$ in **Theorem 10** we get:

Corollary 11 Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there is $c \in (a, b)$, such that

$$\int_a^b f(x)\,dx = f(c) \cdot (b - a).$$

It is not hard to prove that

Theorem 12 Let $f : [a, b] \to \mathbb{R}$ be a function and $c \in (a, b)$. Then f is integrable on $[a, b]$ if and and only if f is integrable on $[a, c]$ and $[c, b]$. In this case we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

proof exercise.

For completeness we define

Definition 13 Let $f : [a, b] \to \mathbb{R}$ be an integrable function $c \in (a, b)$. Then we set

$$\int_c^c f(x)\,dx = 0 \quad \text{and} \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

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Chapter 5.3 - Fundamental theorem of calculus

Aim: We prove the **Fundamental theorem of calculus (FTC)** and then the integration rules. This important theorem is due to Isaac Barrow (1674), Isaac Newton and Gottfried Leibniz.

The FTC can be proven using the **Mean value theorem of integration**. To this end we first consider one of the integration boundaries as a variable.

Theorem 1 (Fundamental theorem of calculus) Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $x \in [a, b]$. Let $F : [a, b] \to \mathbb{R}$ be the function defined by

$$F(x) := \int_a^x f(t)\,dt. \quad \text{Then } F \text{ is differentiable and} \quad \boxed{F'(x) = f(x)} \quad \text{for all} \quad x \in [a, b].$$

Figure We interpret the difference quotient of F in terms of the area under the function f.

proof Idea: We look at the difference quotient for F and then use the **MVT of integration**. We know that for fixed $h \neq 0$ ($h < 0$ possible)

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \cdot \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \frac{1}{h} \cdot \int_x^{x+h} f(t)\,dt. \qquad (*)$$

By the **MVT of integration** there is c_h between x and $x+h$, such that $\int_x^{x+h} f(t)\,dt = h \cdot f(c_h)$. Using this fact and taking the limit in (*) we obtain

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = \lim_{h\to 0} \frac{h}{h} f(c_h) = \lim_{h\to 0} f(c_h) = f(x).$$

This is true as c_h lies between x and $x + h$. Hence F is differentiable with $F' = f$.

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Definition 2 (Primitives) A differentiable function $F:[a,b]\to\mathbb{R}$ is called a **primitive** of a function $f:[a,b]\to\mathbb{R}$ if $F'(x)=f(x)$ for all $x\in\mathbb{R}$.

Theorem 3 Let $F:[a,b]\to\mathbb{R}$ be a primitive of $f:[a,b]\to\mathbb{R}$. Then G is another primitive of f if and only if

$$G=F+c \ \text{ for some constant } \ c\in\mathbb{R}\,.$$

proof " $\Rightarrow$ " If G is another primitive of f then

$$(F-G)'=f-f=0 \ \text{ hence } \ F-G=c \ \text{ for some } \ c\in\mathbb{R}\,.$$

This follows from **Lecture 21, Theorem 7**c).

" $\Leftarrow$ " If $G=F+c$ then $G'=F'=f$. Hence G is also a primitive of f.

Theorem 4 Let $f:[a,b]\to\mathbb{R}$ be a continuous function and F be a primitive of f. Then

$$\int_a^b f(t)\,dt=F(b)-F(a).$$

proof Idea: We use **Theorem 3**. We compare the "standard" primitive $G(x)=\int_a^x f(t)\,dt$ with F. We know

$$G(b)=\int_a^b f(t)\,dt \quad \text{and} \quad G(a)=\int_a^a f(t)\,dt=0.$$

Furthermore $F(x)=G(x)+c$ for all $x\in\mathbb{R}$. Hence

$$F(b)-F(a)=(G(b)+c)-(G(a)+c)=G(b)-G(a)=G(b)=\int_a^b f(t)\,dt.$$

Example: For $k\in\mathbb{Z}$, find a primitive of $\sin(kx)$ and calculate $\int_0^\pi \sin(kt)\,dt$.

Solution: We know that $F(x)=-\frac{\cos(kx)}{k}$ is a primitive of $\sin(kx)$ as $F'(x)=\sin(kx)$. Hence by **Theorem 4** we have

$$\int_0^\pi \sin(kt)\,dt=-\frac{\cos(kx)}{k}\Big|_0^\pi=\frac{-\cos(k\pi)+1}{k}=\begin{cases}0 & \text{if } k \text{ even}\\ \frac{2}{k} & \phantom{\text{if }} k \text{ odd}\,.\end{cases}$$
