Math 35: Real Analysis
Winter 2018

Monday 02/26/18

Lecture 23

Chapter 5 - Integration

Chapter 5.1 - Integration for step functions

Plan: 1.) We define integration first for **step functions** ($\simeq$ area).
2.) We say that a function f is **integrable** if it can be "approximated" by step functions.

Definition 1 (Partitions) A **partition** P of an interval $[a, b]$ is a finite sequence of points $P = \{(x_k)_{k=0,..,n}\}$, such that

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

We call an interval (x_k, x_{k+1}) a **subinterval** of the partition P. We call the width w_P of the largest subinterval

$$w_P = \|P\| = \max\{|x_{k+1} - x_k|, \quad \text{where} \quad k \in \{0, 1, 2, \ldots, n-1\}\} \quad \text{the } \textbf{mesh} \text{ or } \textbf{norm} \text{ of } \; P.$$

If for two partitions P_1, P_2 of $[a, b]$ we have that $P_1 \subset P_2$. Then we call P_2 a **refinement** of P_1.

Defintion 2 (Step functions) A function $f : [a, b] \to \mathbb{R}$ is called a **step function** if there is a partition $P = \{(x_k)_{k=0,..,n}\}$ of $[a, b]$, such that f is constant on each subinterval (x_k, x_{k+1}) of the partition P. We denote by $T([a, b])$ the **set of all step functions** on the interval $[a, b]$.

Example 3: On the interval $[0, 10]$ sketch a partition P_1, a refinement P_2 and a step function f for P_1. Then estimate w_{P_1}. Is f also a step function for P_2?

Note 4: If $P = \{(x_k)_{k=0,..,n}\}$ is a partition of $[a, b]$, then we are not interested in the values of the step function f on the points $(x_k)_{k=0,..,n}$ of the partition. This is because for integration it does not matter which values the the function takes on this finite number of points.

Note 5: Let $[a, b]$ an interval and $f : [a, b] \to \mathbb{R}$ be a step function with its partition P_1. If f is not constant on an open subinterval (x_k, x_{k+1}) of another partition $P_2 = \{(x_k)_{k=0,..,n}\}$. Then f is not a step function for P_2.
We can also say that P_2 is **not** a **valid** or **admissible** partition for f.

Example: Sketch a step function f on the interval $[0, 5]$ with an admissible partion P_1 and a non-admissible partion P_2.

Defintion 6 (Integration for step functions) Let $f \in T([a, b])$ be a step function on $[a, b]$ with respect to the partition $P = \{(x_k)_{k=0,..,n}\}$, i.e.

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

Suppose that for $k \in \{0, 1, \ldots, n-1\}$ we have

$$f(x) = c_k \quad \text{for all} \quad x \in (x_k, x_{k+1}) \quad \text{or} \quad f(x)\big|_{x \in (x_k, x_{k+1})} = c_k.$$

Then we define the **integral** of f ($\simeq$ area) on the interval $[a, b]$ by

$$\int_P f(x)\ dx = \int_a^b f(x)\ dx := \sum_{k=0}^{n-1} c_k \cdot (x_{k+1} - x_k) = \sum_{k=0}^{n-1} c_k \cdot \Delta x_k, \quad \text{where} \quad \Delta x_k = (x_{k+1} - x_k).$$

Example: For your step function from **Example 3** estimate the integral $\int_0^{10} f(x)\ dx$.

Math 35: Real Analysis
Winter 2018

Monday 02/26/18

It remains to show that the integral is well-defined. This means that the integral is independent of the chosen admissible partition for f. To show this we first prove the following lemma.

Lemma 7 Let $[a, b]$ be an interval and P_1 and P_2 be two partions of $[a, b]$. Then there is a common **refinement** P of P_1 and P_2, i.e. $P_1 \subset P$ and $P_2 \subset P$.

Proof Idea: We take the union $P = P_1 \cup P_2$. Then clearly for $P_1 = \{(x_k^1)_{k=0,..,n}\}$ and $P_2 = \{(x_k^2)_{k=0,..,m}\}$ we have after (if necessary) eliminating multiple occurences and reordering

$$P = \{t_0, t_1, \ldots, t_l\} = \{x_0^1, x_1^1, \ldots, x_n^1\} \cup \{x_0^2, x_1^2, \ldots, x_m^2\}$$

is a partition of $[a, b]$, such that $P_1 \subset P$ and $P_2 \subset P$.

proof (Integration for step functions is well-defined) Let $f : [a, b] \to \mathbb{R}$ be a step function and $P_1 = \{(x_k)_{k=0,..,n}\}$ be a partition, such that for $k \in \{0, 1, \ldots, n-1\}$ we have

$$f(x) = c_k \quad \text{for all} \quad x \in (x_k, x_{k+1}).$$

Idea: If the statement is true for refinements then we can use **Lemma 7** to conclude that the value is the same for **any** admissible partition.

1.) Let $P = \{(t_k)_{k=0,..,m}\}$ be a refinement of P_1 then

$$\int_{P_1} f(x)\ dx = \sum_{k=0}^{n-1} c_k \cdot (x_{k+1} - x_k) \quad \text{and} \quad \int_P f(x)\ dx = \sum_{l=0}^{m-1} c'_l \cdot (t_{l+1} - t_l)$$

But as $P_1 \subset P$ we know we know that for all points of P between x_k and x_{k+1} i.e $x_k = t_i < t_{i+1} < \ldots < t_{i+b} = x_{k+1}$: $f(x) = c_k$, therefore

$$c_k \cdot (x_{k+1} - x_k) = \sum_{l=0}^{b-1} c_k \cdot (t_{i+l+1} - t_{i+l}) = c_k \cdot (x_{k+1} - x_k).$$

As this is true for any pair of points (x_k, x_{k+1}), we have that

$$\int_{P_1} f(x)\ dx = \int_P f(x)\ dx$$

Hence the integral is the same for a partition and its refinement.

2.) If P_1 and P_2 are two partitions for f then by **Lemma 7** there is always a common refinement P. Hence

$$\int_{P_1} f(x)\ dx = \int_P f(x)\ dx = \int_{P_2} f(x)\ dx$$

and our statement is true for any (admissible) partition of f.

Math 35: Real Analysis
Winter 2018

Monday 02/26/18

You have seen in Linear Algebra that the set of functions $\mathcal{F}([a,b]) = \{f : [a,b] \to \mathbb{R}\}$ is a vector space with addition and scalar multiplication. We also have

Theorem 8 The set $T([a,b]) = \{f : [a,b] \to \mathbb{R}, f \text{ step function}\}$ is a subspace of $\mathcal{F}([a,b])$. This means we have

a) $0 \in T([a,b])$ (zero function in $T([a,b])$)

b) If $f, g \in T([a,b])$ then $f + g \in T([a,b])$.

c) If $c \in \mathbb{R}$ and $f \in T([a,b])$ then $c \cdot f \in T([a,b])$.

proof HW 8. **Hint:** Use **Lemma 7**.

Using **Lemma 7** we can also show:

Theorem 9 (Linearity and monotonicity of the integral)
Let $f, g : [a,b] \to \mathbb{R}$ be two step functions then

a) $\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$.

b) For $c \in \mathbb{R}$ we have that $\int_a^b c \cdot f(x)\ dx = c \cdot \int_a^b f(x)\ dx$.

c) If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\ dx \le \int_a^b g(x)\ dx$.

Note: This means that $\int : T([a,b]) \to \mathbb{R}$ is a linear function between vector spaces.

proof of Theorem 9 Idea: This follows from the corresponding rules for sums and **Lemma 7**. Example: b) with the notation for f as in **Defintion 1** we have

$$\int_a^b c \cdot f(x)\ dx = \sum_{k=0}^{n-1} c \cdot \underbrace{f(x)}_{=c_k}\big|_{x \in (x_k, x_{k+1})} \cdot (x_{k+1} - x_k) = c \cdot \sum_{k=0}^{n-1} c_k \cdot (x_{k+1} - x_k) = c \cdot \int_a^b f(x)\ dx.$$

Exercise 10 For $f, g \in T([a,b]$ show that

$$\text{a)}\quad f \cdot g \qquad \text{b)}\quad |f|^p \quad \text{for} \quad p \in \mathbb{R}^+ \qquad \text{c)}\quad \min\{f,g\} \quad \text{and} \quad \max\{f,g\}$$

are step functions and therefore integrable.

proof HW 8.

Math 35: Real Analysis
Winter 2018

Monday 02/26/18

Chapter 5.2 - Darboux integral for functions

Aim: We say that a function f is **integrable** if it can be "approximated" by step functions.

Defintion 1 (upper and lower Darboux sums) Let $f : [a,b] \to \mathbb{R}$ be a bounded function and $P = \{(x_k)_{k=0,..,n}\}$ be a partition of $[a,b]$,i.e.

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

We define two step functions $f^U, f_L : [a,b] \to \mathbb{R}$ associated to f and P in the following way

$$\begin{aligned} M_k &= \sup\{f(x), x \in (x_k, x_{k+1})\} \quad \text{and} \quad f^U(x) = M_k \quad \text{for all} \quad x \in (x_k, x_{k+1}) \\ m_k &= \inf\{f(x), x \in (x_k, x_{k+1})\} \quad \text{and} \quad f_L(x) = m_k \quad \text{for all} \quad x \in (x_k, x_{k+1}). \end{aligned}$$

If the partition P is important we will write f_P^U for f^U and $f_{L,P}$ for f_L.
Finally the **Darboux sums** of f with respect to P are the integrals

$$\int_P^U f = \int_a^b f^U(x)\ dx \ \textbf{(upper sum)} \qquad \text{and} \qquad \int_{L,P} f = \int_a^b f_L(x)\ dx \ \textbf{(lower sum)}$$

Example Sketch a continuous function f in the interval $[0,10]$. Using a partition P with four points, sketch f^U and f_L and estimate the integrals $\int_P^U f$ and $\int_{L,P} f$.

Using this approximation with step functions we can try to find the "best approximating" f^U and f_L by varying and refining the partition.

Definition 2 (Upper and lower Darboux integral) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. The **Darboux integrals** of f are

$$\begin{aligned} &\overline{\int_a^b} f(x)\ dx = \inf\{\int_P^U f = \int_a^b f_P^U(x)\ dx,\ \ P \text{ partition of } \ [a,b]\} \quad \textbf{(upper integral)} \\ &\underline{\int_a^b} f(x)\ dx = \sup\{\int_{L,P} f = \int_a^b f_{L,P}(x),\ \ P \text{ partition of } \ [a,b]\} \quad \textbf{(lower integral)} \ . \end{aligned}$$

Math 35: Real Analysis
Winter 2018

Monday 02/26/18

Finally we say that a function f is integrable if the upper and lower Darboux integral coincide. This means that the function can be approximated by greater and lower step functions such that the corresponding integrals exists and are equal.

Defintion 3 (integrable functions) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is **(Darboux) integrable** on the interval $[a, b]$ if

$$\overline{\int_a^b} f(x)\ dx = L = \underline{\int_a^b} f(x)\ dx.$$

In this case we write $L = \int_a^b f(x)\ dx$.

Note: For any partition P of $[a, b]$ we have that

$$\int_a^b f_L(x)\ dx \leq \int_a^b f^U(x)\ dx \quad \text{hence} \quad \underline{\int_a^b} f(x)\ dx \leq \overline{\int_a^b} f(x)\ dx.$$

Example 4: Every step function $f \in T([a, b])$ is **Darboux integrable**.

proof Let f be a step function with partition P. Then

$$f_{L,P} = f = f_P^U \quad \text{hence} \quad \int_a^b f_L(x)\ dx = \int_a^b f_P^U(x)\ dx. \quad (*)$$

By **Defintion 2** we have that

$$\int_a^b f_{L,P}(x)\ dx \leq \underline{\int_a^b} f(x)\ dx \leq \overline{\int_a^b} f(x)\ dx \leq \int_a^b f_P^U(x).$$

Therefore by (*) we have equality, i.e $\underline{\int_a^b} f(x)\ dx = \overline{\int_a^b} f(x) = \int_a^b f(x)\ dx$

Example 5: Let $f : [-1, 1] \to \mathbb{R}$ be the function, such that

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & \phantom{\text{if}} \quad x \in \mathbb{Q}\,. \end{cases}$$

Then f is **not** integrable on $[-1, 1]$.
