# Math 35: Real Analysis
# Winter 2018

Friday 02/23/18

## Lecture 22

**Theorem 9 (L'Hôpital's rule)** Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions, such that $f$ and $g$ are differentiable on $(a, b)\backslash\{c\}$. Suppose that

1.) $g'(x) \neq 0$ for all $x \in (a, b)\backslash\{c\}$.

2.) $\lim_{x\to c} f(x) = f(c) = g(c) = \lim_{x\to c} g(x) = 0$.

3.) $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$, meaning that the limit exists.

Then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = L$.

**proof** We note that by the **Mean value theorem** and 2.) we have that for $x \in (a, b)\backslash\{c\}$ there is $\zeta_x$ in $(x, c)$ or $(c, x)$ such that

$$0 \overset{1.)}{\neq} g'(\zeta_x) = \frac{g(x) - g(c)}{x - c} \overset{2.)}{=} \frac{g(x)}{x - c} \overset{x-c\neq 0}{\Rightarrow} g(x) \neq 0. \quad (*)$$

Hence also $g(x) \neq 0$.
Furthermore by **Cauchy's mean value theorem** we know that for $x \in (a, b)\backslash\{c\}$ there is $z_x$ in $(x, c)$ or $(c, x)$ such that

$$\frac{f'(z_x)}{g'(z_x)} = \frac{f(x) - f(c)}{g(x) - g(c)} \overset{2.)}{=} \frac{f(x)}{g(x)}.$$

As $z_x$ lies between $x$ and $c$ we can take the limit and obtain

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(z_x)}{g'(z_x)} \overset{3.)}{=} L.$$

This proves our statement. □

**Examples:** Use **L'Hôpital's rule** to find the following limits:

$$a) \lim_{x\to 0} -\frac{e^{x^2} - 1}{x^2} \qquad \text{b)} \quad \lim_{x\to c} \frac{h(x^2) - h(c^2)}{\sqrt{x} - \sqrt{c}}$$

Here $h : \mathbb{R} \to \mathbb{R}$ is a differentiable function. Find your result in terms of $h'$.

**Exercise 10** Our aim is to show that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \quad \text{for all} \quad x\in\mathbb{R}\,.$$

a) Show that for a fixed $x\in\mathbb{R}$ we have that

$$\lim_{y\to 0}\frac{\ln(1+x\cdot y)}{y} = x.$$

b) Use **Ch. 4.1, exercise 7** to show that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

c) Use part b) to show that

$$\lim_{n\to\infty}\left(1-\frac{x}{n^2}\right)^n = 1.$$