Math 35: Real Analysis
Winter 2018

Wednesday 02/21/18

Lecture 21

Chapter 4.2 - Properties of differentiable functions

Aim: We will prove a number of theorems about differentiable functions.

Definition 1 (Extreme values) Let I be an interval and $f : I \to \mathbb{R}$ be a function on I. Then f has a

a) **(global) maximum** at c if $f(x) \leq f(c)$ for all $x \in I$.

b) **(global) minimum** at c if $f(x) \geq f(c)$ for all $x \in I$.

c) **(global) extremum** at c if it has either a maximum or minimum at c.

An extremum can also be local. If there is a $\boxed{\delta > 0}$, such that for all $\boxed{x \in (c - \delta, c + \delta)}$

d) $f(x) \leq f(c)$, then f has a **(local) maximum** at c.

e) $f(x) \geq f(c)$, then f has a **(local) minimum** at c.

c) We say that f has a **(local) extremum** at c if it has either a local maximum or local minimum at c.

Note: The maximum and minimum is the value of the function $f(c)$, not the point c.

Example: Sketch a function with both global and local maxima and minima, where one local maximum is not the global maximum and label the extrema. Can a strictly monotone function have a global maximum? Where does the function $f(x) = 1$ have its maxima and minima?

We recall

Definition Let $f : (a, b) \to \mathbb{R}$ be a function and $c \in (a, b)$. Then the function f is **differentiable** at c if the limit

$$\lim_{x \to c} \boxed{\frac{f(x) - f(c)}{x - c}} = f'(c) \qquad \text{exists.}$$

The value $f'(c)$ the **derivative** of f at c.

Definition 2 (critical point) Let I be an interval and $f : I \to \mathbb{R}$ be a function on I. Then c is a **critical point** of f if

$$f'(c) = 0 \quad \text{or} \quad f'(c) \;\; \text{does not exist.}$$

Theorem 3 Let I be an interval and $f : I \to \mathbb{R}$ be a function on I. If f has a local extremum at $x = c$ then c is a critical point of f.

proof If f has a local extremum at $x = c$ and is not differentiable at c then it is a critical point. Suppose that $f(x)$ has a local minimum at $x = c$ and is differentiable. Then there is a δ-neighborhood of c, such that

$$f(c) \leq f(x) \quad \text{for all} \quad x \in (c - \delta, c + \delta).$$

Looking at $F_c(x) = \frac{f(x) - f(c)}{x - c}$ on $(c - \delta, c + \delta)$. For $x < c$ and $x > c$ we get

By the same reasoning $f'(c) = 0$ if f has a local maximum.

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Theorem 4 (Rolle's theorem) Let $f : [a, b] \to \mathbb{R}$ be a continuous function, such that f is differentiable on (a, b). If $f(a) = f(b)$ then there is a point $c \in (a, b)$, such that $f'(c) = 0$.

Example: Sketch a function f on $[0, 5]$ with $f(0) = f(5) = 1$. Then find the point c for your function.

proof If f is constant ($f(x) = f(a)$ for all $x \in [a, b]$) then our statement is obvious. So we may assume that f is not constant. As the interval is closed it follows from the **Extreme value theorem** for continuous functions that f has a minimum m and a maximum M on $[a, b]$. We know that

$$m \leq f(x) \leq M \quad \text{for all} \quad x \in [a, b].$$

If both are equal to $f(a) = f(b)$ then our function would be constant, hence either m or M are different from $f(a) = f(b)$. We may assume that $M = f(c) \neq f(a)$. Then by **Theorem 3** we know that $f'(c) = 0$. □

With the help of **Rolle's theorem** we can prove the following theorem:

Theorem 5 (Mean value theorem - MVT) Let $f : [a, b] \to \mathbb{R}$ be a continuous function, such that f is differentiable on (a, b). Then there is a point $c \in (a, b)$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example Sketch a function f on $[0, 5]$ with $f(0) = 1$ and $f(5) = 5$. Then find the point c from the mean value theorem for this function. What does the theorem mean geometrically?

proof This is a special case of **Rolle's theorem** applied to the function

$$g(x) = (f(x) - f(a)) - \left(\frac{f(b) - f(a)}{b - a}\right) \cdot (x - a).$$

Examples: (Inequalities from the MVT)
a) Setting $b = x$ and $a = 0$ in **Theorem 5** show that

$$\sin(x) \leq x \quad \text{for} \quad x \geq 0 \quad \text{and} \quad \sin(x) \geq x \quad \text{for} \quad x \leq 0.$$

b) Show that for $x > 0$ we have $\frac{x}{x+1} \leq \ln(x+1) \leq x$.
Hint: Use **Theorem 5** with $x = b$. What should be a?

Corollary 6 Let $f : [a, b] \to \mathbb{R}$ be a continuous function, such that f is differentiable on (a, b), such that $m \leq f'(c) \leq M$ for all $c \in (a, b)$. Then

$$m \cdot (y - x) \leq f(y) - f(x) \leq M \cdot (y - x) \quad \text{for all} \quad x, y \in (a, b), y > x.$$

proof HW 7

Another important application of the **Mean value theorem** is the following theorem.

Theorem 7 (Montonicity) Let $f : [a, b] \to \mathbb{R}$ be a continuous function, such that f is differentiable on (a, b). If

a) $f' > 0$ on (a, b) then f is **strictly increasing** on $[a, b]$.

b) $f' \geq 0$ on (a, b) then f is **increasing** on $[a, b]$.

c) $f' = 0$ on (a, b) then f is **constant** on $[a, b]$.

d) $f' \leq 0$ on (a, b) then f is **decreasing** on $[a, b]$.

e) $f' < 0$ on (a, b) then f is **strictly decreasing** on $[a, b]$.

proof a) If $f'(c) > 0$ for all c in (a, b) then by the **Mean value theorem** we have for all $y > x$:

Example: Show that the differential equation $f(x) = f'(x)$ for all $x \in \mathbb{R}$ with $f(0) = A$ has exactly one solution.
Hint: Look at the function $F(x) = f(x) \cdot e^{-x}$.

Note: The first derivative test follows from **Theorem 7**.

Theorem 8 (Cauchy's mean value theorem) Let $f, g : [a, b] \to \mathbb{R}$ be two continuous functions, such that f and g are differentiable on (a, b). Then there is a point $c \in (a, b)$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

proof We apply **Rolle's theorem** to the function

$$h(x) = (f(b) - f(a)) \cdot g(x) - (g(b) - g(a)) \cdot f(x).$$