Math 35: Real Analysis
Winter 2018

Monday 02/19/18

Lecture 20

Chapter 4 - Differentiation

Chapter 4.1 - Derivative of a function

Result: We define the derivative of a function in a point as the limit of a new function, the limit of the **difference quotient**.

Note: Recall that for fixed c and x we have that $\boxed{\dfrac{f(x)-f(c)}{x-c}}$ is the slope of the **secant line** passing through $(c, f(c))$ and $(x, f(x))$.

Definition 1 Let $f : (a,b) \to \mathbb{R}$ be a function and $c \in (a,b)$. Then the function f is **differentiable** at c if the limit

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = f'(c) \qquad \text{exists.}$$

We call the value $f'(c)$ the **derivative** of f at c. The function f is differentiable on (a,b) if it is differentiable at each point of (a,b).

Example: Consider the function $\boxed{f(x) = x^2 + 2x + 1 = (x+1)^2}$.

a) Sketch $f(x)$ in the interval $[-3, 5]$.

b) Draw the secant line through $(1, f(1))$ and $(3, f(3))$ and calculate its slope.

c) Using the definition of the limit calculate $f'(1)$ and $f'(c)$ in general. Then draw the tangent line for $f(x)$ in $x = 1$.

Theorem 1 (f differentiable $\Rightarrow$ f continuous) Let $f : (a, b) \to \mathbb{R}$ be a differentiable function on (a, b). Then f is continuous on (a, b).

proof: We have to show that $\lim_{x\to c} f(x) = f(c)$. We write $f(x) - f(c) = (x - c) \cdot \frac{f(x)-f(c)}{x-c}$. Then

As the derivative is the limit of the function $\boxed{F_c(x) = \dfrac{f(x) - f(c)}{x - c}}$ at $\boxed{\text{x=c}}$ all theorems about limits of functions apply.

Example Calculate $F_1(x)$ for the previous example, the function $f(x) = (x + 1)^2$. What does $F_1(x)$ represent?

Especially we have:

Theorem 3 (Sequence criterion for the derivative) Let $f : (a, b) \to \mathbb{R}$ be a function and $c \in (a, b)$.

a) The function f is differentiable at c if and only if for **any** sequence $(x_n)_n \subset (a, b)\backslash\{c\}$, such that

$$\lim_{n\to\infty} x_n = c \quad \text{we have} \quad \lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} = L = f'(c).$$

b) If $(x_n)_n, (z_n)_n \subset (a, b)\backslash\{c\}$ are two sequences, such that

$$\lim_{n\to\infty} x_n = c = \lim_{n\to\infty} z_n \quad \text{but} \quad \lim_{n\to\infty} \frac{f(x_n) - f(c)}{x_n - c} = L_1 \neq L_2 = \lim_{n\to\infty} \frac{f(z_n) - f(c)}{z_n - c}.$$

Then f is not differentiable in c.

Example: Use **Theorem 3** b) to show that the function given by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if} \quad x = \frac{1}{n} \text{ for } n \in \mathbb{Z} \\ 0 & \quad \text{otherwise} \end{cases} \qquad \text{is not diffferentiable in } \ x = 0.$$

Similarly the linearity of the derivative follows from the limit laws for functions:

Theorem 4 (Linearity of differentiation) Let $f, g : (a, b) \to \mathbb{R}$ be two differentiable functions on (a, b) and $c \in (a, b)$. Then for any constant $k \in \mathbb{R}$ the functions $k \cdot f$ and $f + g$ are differentiable on (a, b) and

$$(k \cdot f)'(c) = k \cdot f'(c) \quad \text{and} \quad (f + g)'(c) = f'(c) + g'(c).$$

proof

For the product and chain rule, however, we have to work a bit:

Theorem 5 (Product rule) Let $f, g : (a, b) \to \mathbb{R}$ be two differentiable functions on (a, b) and $c \in (a, b)$. Then

$$(f \cdot g)'(c) = f'(c) \cdot g(c) + g'(c) \cdot f(c).$$

proof Idea: We use the $a - a = 0$ trick in the limit definition.

Theorem 6 (Chain rule) Let $g : (a_1, b_1) \to \mathbb{R}$ and $f : (a_2, b_2) \to \mathbb{R}$ be two differentiable functions on the given intervals, such that $g((a_1, b_1)) \subset (a_2, b_2)$. Let $c \in (a_1, b_1)$, then $g(c) \in (a_2, b_2)$ and we have

$$(f \circ g)'(c) = f'(g(c)) \cdot g'(c).$$

proof We use the fact that the derivative is the limit of the **difference quotient** and apply this to f. Set

$$F_p(y) = \begin{cases} \frac{f(y)-f(p)}{y-p} & \text{if} \quad y \neq p \\ f'(p) & \quad\quad y = p. \end{cases}$$

Then F_p is continuous at p as f is differentiable in p.

$$\lim_{y \to p} F_p(y) = \lim_{y \to p} \frac{f(y) - f(p)}{y - p} = f'(p). \quad (*)$$

Now we use the $\frac{a}{a} = 1$ trick in the definition of the derivative of $f \circ g$:

$$\begin{aligned} (f \circ g)'(c) &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} = \lim_{x \to c} \left(\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \right) \cdot \left(\frac{g(x) - g(c)}{x - c} \right) \\ &\overset{y=g(x), p=g(c)}{=} \lim_{x \to c} F_{g(c)}(g(x)) \cdot \left(\frac{g(x) - g(c)}{x - c} \right) = \lim_{x \to c} F_{g(c)}(g(x)) \cdot \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= F_{g(c)}(g(c)) \cdot g'(c) = f'(g(c)) \cdot g'(c). \end{aligned}$$

Here

$$\lim_{x \to c} F_{g(c)}(g(x)) = F_{g(c)}(g(c)) = f'(g(c))$$

as g is continuous and $F_{g(c)}$ is continuous by (*). Hence by **Lecture 17, Theorem 5** the function $F_{g(c)} \circ g$ is also continuous and our result follows.

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Example: (Two horrible continuous functions) Consider the functions

$$f(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \phantom{\text{if }} x = 0 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} x^2 \cdot \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \phantom{\text{if }} x = 0 \end{cases}.$$

a) Sketch $f(x)$ and $g(x)$ near $x = 0$.

b) Use the limit definition of the derivative to show that $f(x)$ is not differentiable in $x = 0$. Then show that, however, $g(x)$ is differentiable in $x = 0$.

c) Calculate $g'(x)$ for $x \neq 0$ using the chain and product rule and show that $g'(x)$ is not continuous in $x = 0$, even though $g'(0)$ exists.