Math 35: Real Analysis
Winter 2018

Friday 02/16/18

Lecture 19

Chapter 3.4 - Uniform continuity

Result: Uniform continuity is a stronger version of continuity. Functions that are uniformly continuous have a number of nice properties.

Example 1 Recall that by HW 6 for $f : [2, +\infty) \to \mathbb{R}, x \mapsto f(x) = x^2$ is continuous in any point c and satisfies for all $\epsilon > 0$ there is $\delta > 0$, such that for all $x \in (c-1, c+1)$ we have

$$|x - c| < \frac{\epsilon}{2c+1} = \delta(\epsilon) \Rightarrow \underbrace{|x - c|}_{<\frac{\epsilon}{2c+1}} \cdot \underbrace{|x + c|}_{<2c+1} = |x^2 - c^2| < \epsilon$$

Figure Sketch the function f. Then for $\epsilon = \frac{1}{2}$ write down $\delta(\epsilon)$ for $c = 2$ and $c = 10$ and $c = 100$. What do you see? What happens to $\delta(\epsilon)$ if c increases.

Example 2 The function $g : [1, +\infty) \to \mathbb{R}, x \mapsto g(x) = 2x + 1$ is continuous in any point c. Sketch the function g. Then for the $\epsilon - \delta$ statement find the $\delta = \delta(\epsilon) > 0$ for a given $\epsilon > 0$. Does δ depend on c?

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A function is uniformly continuous on (a, b) if we can find a δ that does not depend on the point c in (a, b).

Definition 3 Let $(a, b) \subset \mathbb{R}$ be an interval. Then the function $f : (a, b) \to \mathbb{R}$ is **uniformly continuous** if for all $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$, such that for all $x, c \in (a, b)$

$$|f(x) - f(c)| < \epsilon \quad \text{for all} \quad |x - c| < \delta.$$

Note: Clearly uniform continuity implies continuity. However the other direction is **not** true.

Example 1: Let $f : [2, +\infty) \to \mathbb{R}, x \to f(x) = x^2$. Take $\epsilon = 1$. We have to show that there are always points x, c, such that for any $\delta > 0$ we have

$$|x - c| < \delta \quad \text{but} \quad |x^2 - c^2| \geq 1$$

The idea is to go further out if δ gets smaller. Take $x = \frac{1}{\delta}$ and $c = \frac{1}{\delta} + \frac{\delta}{2}$. Then

$$|x - c| = \frac{\delta}{2} < \delta \quad \text{but} \quad |x^2 - c^2| = |x - c| \cdot |x + c| = \frac{\delta}{2}\left(\frac{2}{\delta} + \frac{\delta}{2}\right) > 1$$

Theorem 4 If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$. Then f is uniformly continuous on $[a, b]$.

proof
