Math 35: Real Analysis
Winter 2018

Wednesday 02/14/18

Lecture 18

Last time:

Theorem 4 Let $f, g : (a,b) \to \mathbb{R}$ be continuous functions and $c \in (a,b)$. Let $k \in \mathbb{R}$ be a constant. Then

$$k \cdot f \quad , \quad f + g \quad \text{and} \quad f \cdot g$$

are continuous on (a,b). Furthermore if $g(c) \neq 0$ then $\frac{f}{g}$ is continuous in c.

Theorem 5 Let $g : (a,b) \to \mathbb{R}$ be continuous functions and $c \in (a,b)$. Let $f : (u,v) \to \mathbb{R}$ be a continuous function, such that $g((a,b)) \subset (u,v)$. Then

$$f \circ g : (a,b) \to \mathbb{R} \quad \text{is a continuous function.}$$

Note: This means that once we have a collection of basic functions that are continuous, we can prove that more complicated functions are continuous using the previous two theorems.

Example: Given that the function $f(x) = x$ is continuous on $\mathbb{R}$. Explain why the function $x^2 - x + 1$ is continuous. Then explain why any polynomial is continuous on $\mathbb{R}$ and any rational function except for the poles.
Given that $g(x) = \sin(x)$ is continuous on $\mathbb{R}$, find the values in $\mathbb{R}$, where $\frac{1}{\sin(x+1)}$ is continuous.

Note: Without proof we may assume from now on that all known basic functions (rational functions, roots, trig functions ...) are continuous on their domain.

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Definition 6 Let $f:(a,b)\to\mathbb{R}$ or $f:(a,b)\setminus\{c\}\to\mathbb{R}$, be a function, where $c\in(a,b)$.

a) Then f has a **jump discontinuity** at c if

$$\lim_{x\to c^-} f(x) = L_1 \neq L_2 = \lim_{x\to c^+} f(x).$$

b) Then f has a **removable discontinuity** at c if

$$\lim_{x\to c} f(x) = L_1 \neq f(c).$$

Example: Sketch two functions that satisfy **Def. 6** a) and b).

Example: (A horrible function) We can construct functions that are continuous on $\mathbb{R}\setminus\mathbb{Q}$, but discontinuous on $\mathbb{Q}$: Let $f:\boxed{(0,1)}\to\mathbb{R}$ be the function given by

$$f(x) = \begin{cases} 0 & \text{if} \quad x\in\mathbb{R}\setminus\mathbb{Q} \\ \frac{1}{b} & \phantom{\text{if}} \quad x\in\mathbb{Q}, x=\frac{a}{b} \text{ in lowest terms} \end{cases}$$

Show that f is discontinuous for all $c\in\mathbb{Q}$, but continuous for all $c\in\mathbb{R}\setminus\mathbb{Q}$.

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Figure Plot the values of f for all fractions with denominator less or equal to 5.

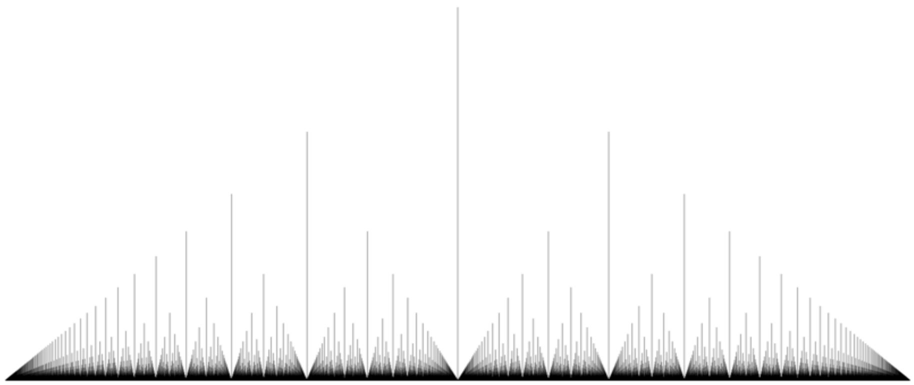

Figure 1: Plot of the Popcorn function $f(x)$.

proof Take $c = \frac{a}{b} \in \mathbb{Q}$ and let $(x_n)_n \subset \mathbb{R} \setminus \mathbb{Q}$ be a sequence of irrational numbers converging to c. Then

$$\lim_{n\to\infty} x_n = 0 \neq f(c) = f\left(\frac{a}{b}\right) = \frac{1}{b}.$$

Hence f is not continuous in c.

However, if $c \in \mathbb{R} \setminus \mathbb{Q}$ then given $\epsilon > 0$, we know that $\epsilon > \frac{1}{n}$ for some $n \in \mathbb{N}$, we can find intervals around c such that

$$\begin{array}{ccccc}
\{\frac{k}{2}, k \in \mathbb{Z}\} \not\subset (c-\delta_2, c+\delta_2) & \Rightarrow & \boxed{f(x) \neq \frac{1}{2}} & \text{for} & x \in (c-\delta_2, c+\delta_2) \\
\{\frac{k}{3}, k \in \mathbb{Z}\} \not\subset (c-\delta_3, c+\delta_3) & \Rightarrow & \boxed{f(x) \neq \frac{1}{3}} & \text{for} & x \in (c-\delta_3, c+\delta_3) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\{\frac{k}{n}, k \in \mathbb{Z}\} \not\subset (c-\delta_n, c+\delta_n) & \Rightarrow & \boxed{f(x) \neq \frac{1}{n}} & \text{for} & x \in (c-\delta_n, c+\delta_n)
\end{array}$$

In total we get that for $\delta = \min\{\delta_l, l \in \{2, 3, \ldots, n\}\}$ by definition of f

$$f(x) = |f(x)| = |f(x) - 0| < \frac{1}{n} = \epsilon \text{ for all } x \in (c-\delta, c+\delta)$$

This means that f is continuous in c.

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Chapter 3.3 - Intermediate and extreme values

Aim: Prove the intermediate value theorem and show that every function on a closed interval has a maximum and a minimum.

Theorem 1 (Intermediate value theorem) Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function. Then for any u between $f(a)$ and $f(b)$ there is

$$c \in [a, b], \quad \text{such that} \quad f(c) = u.$$

Figure: Draw a picture that shows the statement of the theorem.

proof We may assume that $f(a) < u < f(b)$. We have to use the Completeness Axiom. Let

$$S = \{x \in [a, b], f(x) < u\}$$

We know that

Example: Use the intermediate value theorem to show that $f(x) = x - \cos(x)$ has a zero in the interval $[0, \pi]$.

Note: We can use **Theorem 1** to create an algorithm that finds the zeros of continuous function. How would you do that?

Lemma 2 (Continuous functions are bounded) Let $f : [a, b] \to \mathbb{R}$ be continuous function on $[a, b]$. Then f is bounded.

proof By contradicton: Suppose that f is unbounded. Then we show that f can not be continuous on $[a, b]$. We will use the theorem of Bolzano-Weierstrass.

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Theorem 3 (Extreme value theorem) Let $f : [a, b] \to \mathbb{R}$ be continuous function on $[a, b]$. Then f attains its maximum and minimum on $[a, b]$, i.e. there is $c, d \in \mathbb{R}$, such that

$$m = f(c) \leq f(x) \leq f(d) = M \quad \text{for all} \quad x \in [a, b].$$

proof We know by **Lemma 2** that f is bounded. The idea is to use the Bolzano-Weierstrass theorem to show that the infimum and supremum of f on $[a, b]$ is attained.

Note: For the extreme value theorem we have two hypotheses.

1.) The interval must be closed

2.) The interval must be bounded

3.) The function must be continuous

Counterexamples For the following three examples explain which conditions are satisfied and which are not. Then find what the function does not have (max or min).

$$a)\ f(x) = \frac{1}{x-1} \ \text{ on } \ [1, 2] \qquad b)\ g(x) = \frac{1}{x-1} \ \text{ on } \ [2, 3) \qquad c)\ h(x) = x^3 \ \text{ on } \ [0, +\infty).$$
