Math 35: Real Analysis
Winter 2018

Monday 02/12/18

Lecture 17

Last time: Limit of a function at a point:

Definition 1 (Limit of f at the point c) Let $(a, b) \in \mathbb{R}$ be an open interval and $c \in (a, b)$. Let f be a function, such that $f : (a, b) \to \mathbb{R}$ or $f : (a, b)\backslash\{c\} \to \mathbb{R}$.
We say that f **has limit** L **in** c if for all $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{for all} \quad x \in (a, b)\backslash\{c\}, \text{ that satisfy } \quad |x - c| < \delta.$$

In this case we write $\lim_{x\to c} f(x) = L$.

Theorem 2 a) **(Sequence criterion for the limit)** The function f has limit L at c if and only if for **any** sequence $(x_n)_n \subset (a, b)\backslash\{c\}$, such that

$$\lim_{n\to\infty} x_n = c \quad \text{we have} \quad \lim_{n\to\infty} f(x_n) = L.$$

Theorem 3 Let $(a, b) \in \mathbb{R}$ be an open interval and $c \in (a, b)$. Let f be a function, such that $f : (a, b) \to \mathbb{R}$ or $f : (a, b)\backslash\{c\} \to \mathbb{R}$. If $\lim_{x\to c} f(x) = L$ and

$$m \le f(x) \le M \quad \text{for all} \quad x \in (a, b)\backslash\{c\} \quad \text{then} \quad m \le L \le M.$$

proof HW 6

Theorem 4 (Squeeze theorem) Let $(a, b) \in \mathbb{R}$ be an open interval and $c \in (a, b)$. Let f, g, h be functions, such that $f, g, h : (a, b) \to \mathbb{R}$ or $f, g, h : (a, b)\backslash\{c\} \to \mathbb{R}$.
If $\lim_{x\to c} g(x) = L = \lim_{x\to c} h(x)$ and

$$g(x) \le f(x) \le h(x) \quad \text{for all} \quad x \in (a, b)\backslash\{c\} \quad \text{then} \quad \lim_{x\to c} f(x) = L.$$

Example: Use the **Squeeze theorem** to find the limit $\lim_{x\to 0} x^2 \cdot \cos\left(\frac{1}{x^2}\right)$. Then sketch $f(x) = x^2 \cdot \cos\left(\frac{1}{x^2}\right)$ and the bounding functions.

proof of Theorem 4

Note: We can also define a one-sided limit by restricting our definitions to the left-hand or right-hand side of the point c.

Definition 5 (One sided limits) Let $(a, b) \in \mathbb{R}$ be an open interval and $c \in (a, b)$. Let f be a function, such that $f : (a, b) \to \mathbb{R}$ or $f : (a, b)\backslash\{c\} \to \mathbb{R}$.

a) Then f has **right-hand limit** in c if for all $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{for all} \quad x \in (c, b), \text{ that satisfy } \quad |x - c| < \delta.$$

In this case we write $\lim_{x \to c^+} f(x) = L$ or $L = f(c+)$.

b) Then f has **left-hand limit** in c if for all $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{for all} \quad x \in (a, c), \text{ that satisfy } \quad |x - c| < \delta.$$

In this case we write $\lim_{x \to c^-} f(x) = L$ or $L = f(c-)$.

Example: Sketch a function $f(x)$, such that $\lim_{x \to 1^-} f(x) = 1$ and $\lim_{x \to 1^+} f(x) = -1$

Theorem 6 Let $(a, b) \in \mathbb{R}$ be an open interval and $c \in (a, b)$. Let f be a function, such that $f : (a, b) \to \mathbb{R}$ or $f : (a, b) \setminus \{c\} \to \mathbb{R}$. Then

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x).$$

proof

Note: We can also define the limit at infinity. This is the function version of the convergence of a sequence.

Definition 7 (Limit at infinity) Let f be a function, such that $f : (a, \infty) \to \mathbb{R}$.
a)

b) The function f has limit $+\infty$ at infinity if for all $M > 0$ there is $N = N(M) > 0$, such that

$$f(x) > M \quad \text{for all} \quad x \in [N, \infty).$$

We write $\lim_{x \to \infty} f(x) = +\infty$.

b) The function f has limit $-\infty$ at infinity if for all $M > 0$ there is $N = N(M) > 0$, such that

$$f(x) < -M \quad \text{for all} \quad x \in [N, \infty).$$

We write $\lim_{x \to \infty} f(x) = -\infty$.

Note: In a similar fashion we can define the limit at minus infinity. Using an $M - \delta$ definition we can also define $\lim_{x \to c} f(x) = \pm\infty$. It is a good exercise to write down this definition.

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Example: Sketch three functions that describe the situation in **Def. 7** a)-c).

Chapter 3.2 - Continuous functions

Outline: Given the definition of a limit of a function it is easy to define continuity. A function is continuous in a point c if $\lim_{x\to c} f(x) = f(c)$.

Definition 1 Let $f : (a, b) \to \mathbb{R}$ be a function and $c \in (a, b)$. We say that f **is continuous at** c if for all $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$, such that

$$|f(x) - f(c)| < \epsilon \quad \text{for all} \;\; x \in (a, b), \;\text{that satisfy} \;\; |x - c| < \delta.$$

The function is **continuous on** (a, b) if f is continuous at each point of (a, b).

Note 2 Using one-sided limits we can define continuity on closed intervals.

Note 3 The function is continuous in c if and only if $\lim_{x\to c} f(x) = f(c)$. Hence all theorems about limits apply to continuous functions with $L = f(c)$.

It follows directly from **Ch. 3.1.Theorem 3**:

Theorem 4 Let $f, g : (a, b) \to \mathbb{R}$ be continuous functions and $c \in (a, b)$. Let $k \in \mathbb{R}$ be a constant. Then

$$k \cdot f \quad , \quad f + g \quad \text{and} \quad f \cdot g$$

are continuous on (a, b). Furthermore if $g(c) \neq 0$ then $\frac{f}{g}$ is continuous in c.

Theorem 5 Let $g : (a, b) \to \mathbb{R}$ be continuous functions and $c \in (a, b)$. Let $f : (u, v) \to \mathbb{R}$ be a continuous function, such that $g((a, b)) \subset (u, v)$. Then

$$f \circ g : (a, b) \to \mathbb{R} \quad \text{is a continuous function.}$$

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proof Idea: We use the sequence-definition of continuity.
For $c \in (a, b)$ let $(x_n)_n$ be a sequence, such that

$$\lim_{n\to\infty} x_n = c.$$

Then by continuity of g we know that

$$\lim_{n\to\infty} g(x_n) = g(c).$$

Hence the sequence $(g(x_n))_n$ is a sequence that converges to $g(c) \subset (u, v)$. By the continuity of f this implies that

$$\lim_{n\to\infty} f(g(x_n)) = f(g(c)) = f \circ g(c).$$

In total, as our sequence $(x_n)_n$ was chosen arbitrarily, this implies that $f \circ g$ is continuous at c. As this is true for any $c \in (a, b)$, it follows that $f \circ g$ is continuous on (a, b).