Math 35: Real Analysis
Winter 2018

Friday 02/09/18

Lecture 16

Chapter 3 - Continuity

Chapter 3.1 - Limit of a function

Aim: Give a rigorous definition of a limit of a function at a point.

Definition 1 (Limit of f at the point c) Let $(a,b) \in \mathbb{R}$ be an open interval and $c \in (a,b)$. Let f be a function, such that

$$f : (a,b) \to \mathbb{R} \quad \text{or} \quad f : (a,b)\backslash\{c\} \to \mathbb{R}$$

i.e. f might or might not be defined in c.
We say that f **has limit L in c** if for all $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{for all} \quad x \in (a,b)\backslash\{c\}, \text{ that satisfy } \quad |x - c| < \delta.$$

In this case we write $\lim_{x\to c} f(x) = L$. The function **has a limit in c**, if such a number L exists. Otherwise we say that the limit does not exist or f does not have a limit at c.

Note: The limit depends on the values of f near c. The function does not even have to be defined in c.

Example: Consider the function given by

$$f : (0,2)\backslash\{1\} \to \mathbb{R}, x \mapsto f(x) = 2 \cdot \frac{x^2 - 1}{x - 1}$$

1.) Simplify $f(x)$ and then plot the function. What should be the limit $L = \lim_{x\to 1} f(x)$?
2.) Find $\delta(\frac{1}{8})$ and $\delta(\frac{1}{2})$. Is it sufficient to make a single calculation?
Then show that f has limit L. Finally, include into your plot a picture explaining **Def. 1**.

Theorem 2 Let $(a, b) \in \mathbb{R}$ be an open interval and $c \in (a, b)$. Let f be a function, such that

$$f : (a, b) \to \mathbb{R} \quad \text{or} \quad f : (a, b) \backslash \{c\} \to \mathbb{R}$$

i.e. f might or might not be defined in c.

a) The function f has limit L at c if and only if for **any** sequence $(x_n)_n \subset (a, b)\backslash\{c\}$, such that

$$\lim_{n\to\infty} x_n = c \quad \text{we have} \quad \lim_{n\to\infty} f(x_n) = L.$$

b) If $(x_n)_n, (z_n)_n \subset (a, b)\backslash\{c\}$ are two sequences, such that

$$\lim_{n\to\infty} x_n = c = \lim_{n\to\infty} z_n \quad \text{and} \quad \lim_{n\to\infty} f(x_n) = L_1 \neq L_2 = \lim_{n\to\infty} f(z_n).$$

Then f has no limit in c.

Example: Plot $\sin(\frac{1}{x})$ near $x = 0$. Then use part b) to show that $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist.

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proof of Theorem 2 a) " $\Rightarrow$ " Idea: If the $\epsilon - \delta$ statement is true, then it is true for all sequences in the given intervals.
Suppose that $\lim_{x \to c} f(x) = L$. So we know: For all $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{for all} \ \ x \in (a,b) \backslash \{c\}, \ \text{that satisfy} \ \ |x - c| < \delta. \quad (*)$$

Let $(x_n)_n$ be a sequence, such that $\lim_{n \to \infty} x_n = c$. Using the ϵ'- definition of convergence for sequences (we need a different ϵ) with $\epsilon' = \delta$ we know that there is $N(\delta) \in \mathbb{N}$, such that

" $\Leftarrow$ " The other direction is a bit trickier. So we know that for any sequence $(x_n)_n$ in $(a,b) \backslash \{c\}$, such that

$$\lim_{n \to \infty} x_n = c \ \text{ we have } \ \lim_{n \to \infty} f(x_n) = L.$$

We have to show that for all $\epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$, such that

$$|f(x) - L| < \epsilon \quad \text{for all} \ \ x \in (a,b) \backslash \{c\}, \ \text{that satisfy} \ \ |x - c| < \delta. \quad (**)$$

Suppose that the $\epsilon - \delta$ statement does not hold. Then there is $\epsilon > 0$, such that there is **no** $\delta = \delta(\epsilon) > 0$, such that (**) is true.
This means that for this ϵ we know that for every $\delta > 0$ there is at least one $x' \in (a,b) \backslash \{c\}$, such that

Theorem 2 part b) is a logical consequence of part a):

From **Theorem 2** a) we conclude directly:

Theorem 3 Let $f, g : (a,b) \to \mathbb{R}$ or $f, g : (a,b)\backslash\{c\} \to \mathbb{R}$ be two functions, such that

$$\lim_{x\to c} f(x) = L_1 \quad \text{and} \quad \lim_{x\to c} g(x) = L_2.$$

Then we have

a) For all $s \in \mathbb{R}$ we have: $\lim_{x\to c} s \cdot f(x) = s \cdot \lim_{x\to c} f(x)$.

b) $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L_1 + L_2$.

c) $\lim_{x\to c}(f(x) \cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = L_1 \cdot L_2$.

d) If $g(x) \neq 0$ for all $x \in (a,b)\backslash\{c\}$ and $L_2 \neq 0$ then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} f(x) \cdot \lim_{x\to c} \frac{1}{g(x)} = \frac{L_1}{L_2}.$$

proof