Math 35: Real Analysis
Winter 2018

Wednesday 02/07/18

Lecture 15

Chapter 2 - Review: Sequences and series

Fibonacci sequence : Let $(F_n)_n$ be the sequence given by $F_1 := 1$, $F_2 = 1$ and

$$F_{n+1} = F_n + F_{n-1} \quad \text{for all} \quad n \geq 3.$$

Aim: Find a closed-form expression for F_n.

We have seen in **Lecture 8**, example 2.c) that
1.)$F_n \geq 1$ for all $n \in \mathbb{N}$ and $F_n < F_{n+1}$ for all $n \geq 2$ i.e. $(F_n)_n$ is strictly increasing.

2.) Let $(R_n)_n$ be the sequence of ratios of consecutive Fibonacci numbers, i.e.

$$R_n := \frac{F_{n+1}}{F_n} \quad \text{for all} \quad n \in \mathbb{N}$$

As $F_n \geq 1$ this is well-defined. By 1.) We also have that $F_n \leq F_{n+1} \Rightarrow \boxed{1 \leq R_n}$.

3.) We note that

$$\boxed{R_n} = \frac{F_{n+1}}{F_n} \overset{\text{Def.}F_n}{=} \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} = \boxed{1 + \frac{1}{R_{n-1}}}.$$

4.) If the limit L exists, then

$$\boxed{L} = \lim_{n\to\infty} R_n \overset{3.)}{=} \lim_{n\to\infty} 1 + \frac{1}{R_{n-1}} \overset{\textit{limit laws}}{=} 1 + \frac{1}{\lim_{n\to\infty} R_{n-1}} = \boxed{1 + \frac{1}{L}}.$$

Hence

$$L = 1 + \frac{1}{L} \Leftrightarrow L^2 - L - 1 = 0 \Leftrightarrow \quad (L = \frac{1+\sqrt{5}}{2} = \phi) \quad \text{or} \quad (L = \frac{1-\sqrt{5}}{2} = \psi).$$

If the sequence converges, the only possible limits are ϕ or ψ. As by 2.) $L \geq 1$ the only option is

$$L = \phi = \frac{1+\sqrt{5}}{2} \simeq 1.61803398875... \quad \textbf{(Golden ratio)}.$$

5.) $(R_n)_n$ converges to ϕ as

$$|R_n - \phi| \overset{3.),4.)}{=} \left|\left(1+\frac{1}{R_{n-1}}\right) - \left(1+\frac{1}{\phi}\right)\right| = \left|\frac{1}{R_{n-1}} - \frac{1}{\phi}\right|$$
$$= \left|\frac{\phi - R_{n-1}}{\phi \cdot R_{n-1}}\right| \overset{\frac{1}{R_{n-1}} \leq 1 \; by\, 2.)}{\leq} \frac{1}{\phi} \cdot |R_{n-1} - \phi|$$

Hence by induction we can show that $0 \leq |R_n - \phi| \leq \left(\frac{1}{\phi}\right)^{n-1} \cdot |R_1 - \phi|$.
As $\phi > 1 \Rightarrow \frac{1}{\phi} < 1$ it follows by the **Squeeze theorem** that

$$0 = \lim_{n\to\infty} 0 = \lim_{n\to\infty} |R_n - \phi| = \lim_{n\to\infty} \left(\frac{1}{\phi}\right)^{n-1} \cdot |R_1 - \phi| = 0.$$

This is equal to $\lim_{n\to\infty} R_n = \phi$. Hence $(R_n)_n$ converges to ϕ.

6.) Consider the series $F(x) = \sum_{k=1}^{\infty} F_k \cdot x^k$.
Find an interval $(a, b) \in \mathbb{R}$, such that the series converges for all $x \in (a, b)$.

7.) Using the definition of $(F_n)_n$ show that $F(x) = x + x \cdot F(x) + x^2 \cdot F(x)$ for all $x \in (a, b)$.

8.) By 7.) we have that

$$F(x) = \frac{x}{1 - x - x^2}.$$

Factor the denominator of the above expression and then express $F(x)$ as the sum of two simple rational functions.

9.) Express the two simple rational functions from 8.) as two power series.

10.) Compare the coefficients from 9.) with the ones from 6.) to find the closed-form expression for F_n.