Math 35: Real Analysis
Winter 2018

Monday 02/05/18

Lecture 14

Chapter 2.4 - Subsequences

Outline: Every sequence of real numbers has a monotone subsequence. This implies that every bounded sequence has a convergent subsequence.

Definition 1 (Subsequence) Let $(a_n)_n$ be a sequence and $(k_n)_n \subset \mathbb{N}$ be a **strictly** increasing sequence of natural numbers. Then the sequence $(a_{k_n})_n$ is called a **subsequence** of $(a_n)_n$.

Idea: This means that a subsequence can skip values of the original sequence $(a_n)_n$, but can never repeat a value of $(a_n)_n$.

Example of $(k_n)_n$:

Example: Let $(a_n)_n$ be the sequence given by $a_n := \frac{1}{n^2}$ and $(k_n)_n$ the sequence given by $k_n := 2n+1$. Write down the first six elements of $(a_n)_n$, the first four elements of $(k_n)_n$ and the first four elements of $(a_{k_n})_n$. Does $(a_{k_n})_n$ converge? What is the limit?

Theorem 2 Let $(a_n)_n$ be a sequence. Then

a) If $(a_n)_n$ converges to a then every subsequence of $(a_n)_n$ converges to a.

b) If $(a_n)_n$ has two subsequence that converge to different limits, then $(a_n)_n$ does not converge.

proof: a) By the ϵ-criterion for convergence we have: For all $\epsilon > 0$ there is $N = N(\epsilon) \in \mathbb{N}$, such that

$$|a_n - a| < \epsilon \quad \text{for all} \quad n \geq N \quad (*)$$

Let $(a_{k_n})_n$ be a subsequence. As $(k_n)_n \in \mathbb{N}$ is a strictly increasing sequence we know that

$$k_1 < k_2 < k_3 < \ldots < k_n < \ldots$$

Especially $k_{n+1} \geq k_n + 1$ as it is a sequence of natural numbers. Therefore we must have that

$$k_n \geq n \quad \text{for all} \quad n \in \mathbb{N}\,.$$

Hence as $k_n \geq n$ by (*) we also have that

$$|a_{k_n} - a| < \epsilon \quad \text{for all} \quad n \geq N \quad (*)$$

Therefore the subsequence converges to a.

b) Idea: We show that in this case $(a_n)_n$ can not be a Cauchy sequence, therefore it can not converge.

Figure: Sketch the sequence $(2 \cdot (-1)^n)_n$ and two subsequences that converge to different limits.

Let $(a_{k_n})_n$ and $(a_{l_n})_n$ be two subsequences of $(a_n)_n$, such that

$$\lim_{n\to\infty} a_{k_n} = a \quad \text{and} \quad \lim_{n\to\infty} a_{l_n} = b.$$

Then these two points have a certain distance $|b - a|$. Take $\epsilon = \frac{|b-a|}{4}$. We know that there is N_1 and N_2 in $\mathbb{N}$, such that

$$|a_{k_n} - a| < \epsilon \quad \text{for all} \quad n \geq N_1 \quad \text{and} \quad |a_{l_m} - a| < \epsilon \quad \text{for all} \quad m \geq N_2$$

Hence for all $n, m \geq N = \max\{N_1, N_2\}$ we have that

$$|a_{k_n} - a_{l_m}| > \frac{|b-a|}{2}$$

Hence for $\tilde{N} \geq \max\{k_N, l_N\}$ there are always pairs of terms in $(a_n)_n$ whose distance is greater than $\frac{|b-a|}{2}$. This means that $(a_n)_n$ is not a Cauchy sequence and therefore does not converge.

Theorem 3 Every sequence $(a_n)_n$ has a monotone subsequence.

Example: Sketch the sequence $(\frac{\cos(n)}{n})_n$ and identify an increasing subsequence. Can you also find a decreasing subsequence?

proof: Idea: We try to construct an increasing subsequence. If our construction fails we show that there is a decreasing subsequence.
We let S be the set of integers $n \in \mathbb{N}$, such that a_n is a lower bound for $\{a_n, a_{n+1}, \ldots\}$ i.e.

$$a_n \leq a_k \quad \text{for all} \quad k \geq n+1$$

Example: From your previous sketch identify the first elements in S for the sequence $(\frac{\cos(n)}{n})_n$ and mark the corresponding terms of the sequence.

If S is infinite, i.e. $S = \{k_1, k_2, k_3, \ldots\}$ then the subsequence $(a_{k_n})_n$ is a strictly increasing sequence by the definition of S. Hence we have found an increasing subsequence.

If S is finite, then there is $N \in \mathbb{N}$, such that N is larger than any element in S.
Using this fact we can construct a decreasing subsequence in the following way:
Set $l_1 = N$ As a_N is not a lower bound for the set $\{a_N, a_{N+1}, \ldots\}$, there is $l_2 > l_1$, such that

$$a_N = a_{l_1} > a_{l_2}.$$

Again, as $N < l_2 \notin S$, a_{l_2} it not a lower bound for the following elements of $(a_n)_n$. Hence there is l_3, such that

$$a_N = a_{l_1} > a_{l_2} > a_{l_3}$$

Continuing this way we find a strictly decreasing subsequence $(a_{l_n})_n$.
In any case we can find a monotone sequence, hence our statement is true. $\square$

Theorem 4 (Bolzano-Weierstrass) Every bounded sequence $(a_n)_n$ has a converging subsequence.