Math 35: Real Analysis
Winter 2018

Monday 01/29/18

Lecture 11

Chapter 2.3 - Infinite series

Outline: Given a sequence $(a_k)_k$ we can make a new sequence $(S_n)_n$, where we obtain S_n by summing up over the first n elements of $(a_k)_k$. The limit $\lim_{n\to\infty} S_n$ is an **infinite series**.

Definition 1 Let $(a_k)_{k\in\mathbb{N}}$ be a sequence then the sequence $(S_n)_{n\in\mathbb{N}}$ defined by

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n.$$

is called a sequence of **partial sums** S_n. No matter if the limit exists or not

$$S_\infty = \sum_{k=1}^{\infty} a_k$$

is called an **(infinite) series**.
If $\lim_{n\to\infty} S_n = S = S_\infty$ exists, then the series is called **convergent**. In this case we write

$$\sum_{k=1}^{\infty} a_k = S.$$

Note: By abuse of notation the sequence of partial sums is sometimes also called the series.

Example: For the following sequences $(a_k)_k$ write down $(S_n)_n$ and calculate the first three terms of $(S_n)_n$. Then decide whether they converge:

$$\text{a)} \quad a_k := k \quad \text{b)} \quad a_k := \frac{1}{k} \quad \text{c)} \quad a_k = \frac{1}{2^k}$$

Figure 2: Draw a picture for c) that shows how a series is related to integration of step functions.

We recall that a sequence $(a_n)_n$ converges if and only if it is a Cauchy sequence i.e. for any $\epsilon > 0$ there is an $N = N(\epsilon)$, such that

$$|a_n - a_m| < \epsilon \quad \text{for all} \quad n, m \geq N.$$

Theorem 3 (Cauchy criterion for series) A series $\sum_{k=1}^{\infty} a_k$ converges if and only if there is an $N = N(\epsilon) \in \mathbb{N}$, such that

$$|S_n - S_m| = |\sum_{k=m+1}^{n} a_k| < \epsilon \quad \text{for all} \quad n \geq m \geq N.$$

Figure 4: For $n \geq m \geq 5$ interpret this theorem in terms of Figure 2.

Theorem 5 (Absolute convergence) If the series $\sum_{k=1}^{\infty} |a_k|$ converges then $\sum_{k=1}^{\infty} a_k$ converges. If $\sum_{k=1}^{\infty} |a_k|$ converges then we say that $\sum_{k=1}^{\infty} a_k$ **converges absolutely**.

proof: We only use the Cauchy criterion and the triangle inequality:
Fix $\epsilon > 0$. As $\sum_{k=1}^{\infty} |a_k|$ converges we know by **Theorem 4** that there is an $N = N(\epsilon) \in \mathbb{N}$, such that

This means that $\left(\sum_{k=1}^{n} a_k\right)_n$ is a Cauchy sequence and therefore convergent. $\square$

Exercise: Show that the series $\sum_{k=1}^{\infty} \frac{1+\cos(k)}{2^k}$ converges.

Theorem 6 If a series $\sum_{k=1}^{\infty} a_k$ converges then $\lim_{k\to\infty} a_k = 0$.

proof: Recall that $S_n = \sum_{k=1}^{n} a_k$ and that the series converges if and only if $(S_n)_n$ converges. Let $S = \lim_{n\to\infty} S_n$ be the limit. We know that $\boxed{a_{n+1} = S_{n+1} - S_n}$ hence

Note: The converse of **Theorem 6** is not true. An example is $\sum_{k=1}^{\infty} \frac{1}{k}$.

Theorem 7 If $a_k \geq 0$ for all $k \in \mathbb{N}$ and $(S_n)_n = \left(\sum_{k=1}^{n} a_k\right)_n$ is bounded above then the series $\sum_{k=1}^{\infty} a_k$ converges.

proof: This is a consequence of **Ch. 2.2. Theorem 1**. In this case $(S_n)_n$ is bounded and increasing.

Theorem 8 (Comparison test) Let $a_k, b_k \geq 0$ for all $k \in \mathbb{N}$. and $(S_n)_n = \left(\sum_{k=1}^{n} a_k\right)_n$ and $(T_n)_n = \left(\sum_{k=1}^{n} b_k\right)_n$ be two series. If for fixed $K \in \mathbb{N}$ and for all $n \geq K$ we have that

a) $\sum_{k=K}^{n} a_k \leq \sum_{k=K}^{n} b_k \leq B$ for some $B \in \mathbb{R}^+$ then $\sum_{k=1}^{\infty} a_k$ converges.

b) $\sum_{k=K}^{n} b_k \leq \sum_{k=K}^{n} a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges then $\sum_{k=1}^{\infty} a_k$ diverges.

proof: HW 5.