Math 35: Real analysis
Winter 2018 - Final exam (take-home)

Total: *50 points* Return date: Monday 03/12/18 at 4pm in KH 318

keywords: *subsequences, uniform continuity, differentiation, integration*

Instructions: Please show your work; no credit is given for solutions without work or justification. No collaboration is permitted on this exam. You may consult the textbook, your lecture notes and the homework, but no other sources (books or internet) are allowed.

problem 1 Let $f : [0,1] \to \mathbb{R}$ be an increasing function, that is **not** continuous. Let $(x_n)_n$ be a sequence of points in $[0,1]$. Show that there is a subsequence $(x_{k_n})_n$ of the sequence $(x_n)_n$, such that $(f(x_{k_n}))_n$ is a converging sequence.

problem 2 Let $f : \mathbb{R} \to \mathbb{R}$ be a function, such that for all $x, y \in \mathbb{R}$

$$|f(y) - f(x)| < 5 \cdot (|y - x|)^{\frac{1}{2}}.$$

Prove that f is uniformly continuous on $\mathbb{R}$.

problem 3 Let $f : [a,b] \to [a,b]$ be a differentiable and therefore continuous function that satisfies: there is $q < 1$, such that

$$|f'(x)| \leq q < 1 \quad \text{for all} \quad x \in [a,b].$$

For a fixed $z \in [a,b]$ we define a sequence $(x_n)_n$. We set

$$x_0 := z, \quad \text{and} \quad \boxed{x_n := f(x_{n-1})} \quad \text{for all} \quad n \geq 1.$$

Show that the sequence $(x_n)_n$ converges to a point ξ, such that $\boxed{f(\xi) = \xi}$ in the following steps.

a) Show that if the limit $\lim_{n\to\infty} x_n = \xi$ exists then we have that $f(\xi) = \xi$.

b) Show that $\left|\frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}\right| \leq q$.

c) Using the result from b) give an upper bound for $|x_{n+1} - x_n|$ in terms of $|x_1 - x_0|$.

d) Using the fact that $x_{n+1} = x_0 + \sum_{k=0}^{n}(x_{k+1} - x_k)$ show that the sequence $(x_n)_n$ is convergent.

e) Show that there is exactly one solution to the equation $f(y) = y$ in the interval $[a,b]$. This means that the limit of the sequence is independent of the starting point $x_0 = z$.

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problem 4 Prove the following theorem:

Theorem (Cauchy-Schwarz inequality for integration) Let $f, g : [a,b] \to \mathbb{R}$ be two continuous functions. Then f^2 and g^2 are continuous functions and we have

$$|\int_a^b f(x)g(x)\,dx|^2 \leq \left(\int_a^b f^2(x)\,dx\right) \cdot \left(\int_a^b g^2(x)\,dx\right)$$

Hint: The proof of **Lecture 24, Theorem 7** shows that we can find approximating lower and upper step functions for f and g on an equidistant partition.

problem 5 Let $f : [a,b] \to \mathbb{R}$ be a **monotone** function.
Show that f is integrable on the interval $[a,b]$.

Hint: Similar to the proof of **Lecture 24, Theorem 7** find lower and upper step functions f_L and f^U for f on an equidistant partition.