Math 35: Real Analysis
Winter 2018

Friday 01/05/18

Axioms for the real numbers

The following axioms define the real numbers completely. They are phrased as in the book.

Definition 1.1 A field is a non-empty set F of objects that has two operations defined on it. These operations are called addition and multiplication and are denoted in the usual way. Addition and multiplication satisfy the following properties:

Addition:

1. For all $x, y \in F, x + y \in F$.

2. For all $x, y \in F, x + y = y + x$. (commutative law)

3. For all $x, y, z \in F, (x + y) + z = x + (y + z)$. (associative law)

4. F contains an element 0 such that $x + 0 = x$ for all $x \in F$. (0 is the neutral element)

5. For all $x \in F$ there is $y \in F$, such that $x + y = 0$. (inverse elements)

Note: Properties **1.-5.** are equivalent to $(F, +)$ being a commutative group.

Multiplication:

6. For all $x, y \in F, x \cdot y \in F$.

7. For all $x, y \in F, x \cdot y = y \cdot x$. (commutative law)

8. For all $x, y, z \in F, (x \cdot y) \cdot z = x \cdot (y \cdot z)$. (associative law)

9. F contains an element $1 \neq 0$ such that $x \cdot 1 = x$ for all $x \in F$. (1 is the neutral element)

10. For all $x \in F \backslash \{0\}$ there is $y \in F \backslash \{0\}$, such that $x \cdot y = 1$. (inverse elements)

Distributive law:

11. For all $x, y, z \in F, (x + y) \cdot z = x \cdot z + y \cdot z$.

Note: Properties **6.-11.** imply that $(F \backslash \{0\}, \cdot)$ is a group.

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Definition 1.2 An **order** $<$ on a set S is a relation that satisfies the following two properties:

1. If $x, y \in S$ then exactly one of the three cases is true: $x < y$, $x = y$, or $x > y$.

2. For all $x, y, z \in S$: If $x < y$ and $y < z$ then $x < z$.

An **ordered set** is a set with an order $<$ defined on it.

Definition 1.3 An **ordered field** is a field F with an order $<$ with the following additional properties:

1. If $x > 0$ and $y > 0$ then $x + y > 0$.

2. If $x > 0$ and $y > 0$ then $x \cdot y > 0$.

3. $x < y$ if and only if $y - x > 0$.

$\mathbb{R}$ is an ordered field that additionally satisfies the **Completeness Axiom**.

Definition 1.14.a) Let $S \subset \mathbb{R}$ be a non-empty set of real numbers. The set S is **bounded above** if there is a number M, such that

$$x \leq M \quad \text{for all} \quad x \in S.$$

The number M is called an **upper bound** of S.

Definition 1.15.a) Let $S \subset \mathbb{R}$ be a non-empty set of real numbers. Suppose S is bounded above. The number β is the **supremum of** S if β is an upper bound of S and any number less than β is not an upper bound of S i.e.

$$\text{for all} \quad b < \beta \quad \text{there is an} \quad x \in S, \quad \text{such that} \quad b < x.$$

We will write $\beta = \sup(S)$.

Completeness Axiom: Each non-empty set $S \subset \mathbb{R}$ of real numbers that is bounded above has a supremum $\sup(S)$.
