28. Let x, y, and z be positive numbers. Find the minimum value for the sum $x^2 + y^2 + z^2$, subject to the condition $xy^2z = 12$.

29. Find the maximum volume for a right circular cylinder if its surface area is a constant value S. Consider the case in which the cylinder has a bottom but no top as well as the case in which it has a top and a bottom.

30. Let n be a positive integer and let $a_1, a_2, \ldots, a_n$ be positive numbers. The **harmonic mean** of these numbers is the reciprocal of the arithmetic mean of the reciprocals of the numbers. Prove that the harmonic mean of a set of positive numbers is less than or equal to the geometric mean. When does equality occur?

31. Let n be a positive integer. Suppose that $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers and that at least one of the b_k's is not zero. For each real number t, let

$$P(t) = \sum_{k=1}^{n} \left(a_k - tb_k\right)^2.$$

   a) Show that P is a polynomial in t of degree 2.
   b) By completing the square, find the value of t that minimizes P.
   c) Show that the value of t from part (b) is the appropriate choice for the constant z that is needed in the proof of the Cauchy-Schwarz Inequality.

32. Prove the Cauchy-Schwarz Inequality for the case $n = 2$ by writing out both sides of the inequality, then multiplying and rearranging terms until a familiar inequality is obtained. Make certain that the steps are reversible!

33. Rephrase the Cauchy-Schwarz Inequality in the language of the vector space $\mathbb{R}^n$.

34. Let $a_1, a_2, \ldots, a_n$ be real numbers. Prove that $\sum_{k=1}^{n} |a_k| \le \sqrt{n}\sqrt{\sum_{k=1}^{n} a_k^2}$.

35. Let r be a fixed positive real number. Suppose that a, b, and c are real numbers that satisfy $a^2 + b^2 + c^2 = r^2$. Find the maximum value of $|a| + |b| + |c|$ and the values of a, b, and c that generate the maximum value.

36. Suppose that $a_1, a_2, \ldots, a_n$ are positive real numbers. Find the minimum value of the expression $\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right)$.

37. Suppose that $x > -1$ and that $x \neq 0$. Prove that $(1 + x)^n > 1 + nx$ for each positive integer $n > 1$. This result is known as **Bernouilli's Inequality**.

1.3 The Completeness Axiom

The rational numbers are closed under addition and multiplication, but the rational numbers are not closed under the process of finding roots. This was illustrated in Section 1.1 with a proof that $\sqrt{2}$ is not a rational number. The rational numbers are also not closed under the limit process since there are convergent sequences of rational numbers that do not converge to a rational number. (We assume that the reader has had a little exposure to the limit process.) For example, the sequence

$$.101,\ .101001,\ .1010010001,\ .101001000100001,\ .1010010001000010000001,\ \cdots$$

is a sequence of rational numbers that converges to an irrational number (the limit is a number whose decimal expansion has no repeating pattern). Since the limit process is central to the development of calculus

8. Prove that a nonempty set that is bounded above has only one supremum.

9. The Completeness Axiom only asserts something about sets that are bounded above. Use the Completeness Axiom to prove that every nonempty set of real numbers that is bounded below has an infimum.

10. Prove that the infimum and supremum of the interval (a, b) are a and b, respectively.

11. Prove that a nonempty finite set contains its infimum.

12. Let S be a nonempty set of real numbers that is bounded above and let $\beta = \sup S$. Prove that for each $\epsilon > 0$ there exists a point $x \in S$ such that $x > \beta - \epsilon$.

13. Find the supremum of the set $\{x : 3x^2 + 3 < 10x\}$.

14. Use the Completeness Axiom to finish the proof of Theorem 1.9, that is, to prove that an interval has one of nine possible forms. There are a number of cases to consider: the set is bounded, the set is bounded above but not below, etc.

15. Let a be a positive number. Prove that for each real number x there is an integer n such that $na \leq x < (n + 1)a$.

16. Referring to Theorem 1.17, prove (1) $\Rightarrow$ (2).

17. Referring to Theorem 1.17, prove (3) $\Rightarrow$ (4).

18. Prove each of the following results—give a direct proof of each one—without use of the Completeness Axiom. These results could be called the Archimedean Property of the rational numbers.

a) If a and b are positive rational numbers, then there exists a positive integer n such that $na > b$.

b) For each positive integer n, there exists a rational number r such that $r > n$.

c) For each rational number x, there exists an integer n such that $n \leq x < n + 1$.

d) For each positive rational number x, there is a positive integer n such that $1/n < x$.

19. Let s and t be real numbers such that $t - s > 1$. Prove that there exists an integer p such that $s < p < t$.

20. Let x be a real number. Prove the following statement: for each $\epsilon > 0$, there exists a rational number r such that $0 < |x - r| < \epsilon$.

21. A real number of the form $p/2^n$, where p is an integer and n is a nonnegative integer, is known as a **dyadic rational number**. Prove that there is a dyadic rational number between any two distinct real numbers.

22. Consider the set A defined in the proof of Theorem 1.19. Prove that b is an upper bound of A if $b^3 > 5$.

23. Use an argument similar to the one in the proof of Theorem 1.19 to prove

a) there is a real number x such that $x^2 = 2$;

b) there is a real number y such that $y^3 = 7$.

24. Prove that every decimal expansion represents a real number.

Remark. The rest of the exercises in this section are in no particular order.

25. Prove that each real number $x \in [0, 1]$ h... $x \in [0, 1]$, prove that there i... is, for each ... either 0 or 1 ...

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b) Use the idea in part (a) to determine an explicit one-to-one correspondence between $\mathbb{Z}^+$ and the set of all ordered pairs of positive integers.

c) Use part (a) to give a different proof of Theorem 1.26.

15. Prove Lemma 1.27.

16. Prove Lemma 1.28.

17. Suppose that a and b are distinct real numbers such that $a < b$.

a) Prove that the set $\{x \in \mathbb{R} : a < x < b\}$ is uncountable.

b) Prove that the set $\{x \in \mathbb{Q} : a < x < b\}$ is countably infinite.

Remark. The rest of the exercises in this section are in no particular order.

18. Let A be the collection of all sequences of 0's and 1's. One example of an element of the set A is the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots.$$

Let B be the subset of A that consists of all sequences of 0's and 1's for which the number of 1's is finite. One example of an element of the set B is the sequence

$$1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots.$$

a) Use Theorem 1.25 to show that the set B is countably infinite.

b) Prove that the set A is uncountable. (The standard proof of this result is known as **Cantor's diagonal process**. Assume that the set A is countably infinite, that is, each positive integer corresponds to one sequence in A. Write out the sequences underneath each other, a first sequence, a second sequence, and so on. The resulting figure looks like a matrix composed of 0's and 1's. Use the diagonal of this matrix to generate a sequence in A that is not in the list.)

c) Prove that the collection of all subsets of positive integers is uncountable by establishing a one-to-one correspondence between this set and the set A.

d) Use Exercise 25 in Section 1.3 and parts (a) and (b) to prove that the interval $[0, 1]$ is uncountable.

19. Use the fact that every real number has a unique decimal expansion that does not end in all 9's to prove that the interval $(0, 1)$ is an uncountable set.

20. Let $\mathcal{F}$ be a collection of disjoint intervals of real numbers. This means that if I and J are two distinct intervals in $\mathcal{F}$, then $I \cap J = \emptyset$. Prove that the set $\mathcal{F}$ is countable.

21. Prove that every infinite set contains a countably infinite subset.

22. Prove that a set is infinite if and only if it can be put into a one-to-one correspondence with one of its proper subsets. (A set A is a proper subset of B if $A \subseteq B$ but $A \neq B$.)

23. The set of real numbers is composed of the rational numbers and the irrational numbers. Another classification of real numbers involves algebraic and transcendental numbers. An **algebraic number** is any number that is a root of a polynomial with integer coefficients. For example, the number $\sqrt{2}$ is algebraic since it is a root of the polynomial $x^2 - 2$. A **transcendental number** is a real number that is not an algebraic number.

a) Prove that every rational number is an algebraic number.

b) Prove that the square root and cube root of every positive integer is an algebraic number.

c) Prove that $2 - \sqrt{3}$ and $\sqrt{\ldots}$