- subject to the condition  $xy^2z = 12$ . subject to the condition and subject to the condition as a right circular cylinder if its surface area is a constant 29. Find the maximum volume for a right circular cylinder has a bottom but no top as well as a constant condition of Consider the case in which the cylinder has a bottom but no top as well as a constant condition of conditions are conditions.
- Find the maximum volume for a light value S. Consider the case in which the cylinder has a bottom but no top as well as the value S. Consider the case in and a bottom. case in which it has a top and a bottom. case in which it has a ver-30. Let n be a positive integer and let  $a_1, a_2, \ldots, a_n$  be positive numbers. The harmonic
- Let n be a positive integer and reciprocal of the arithmetic mean of the reciprocals of mean of these numbers is the reciprocal of a set of positive numbers in mean of these numbers is the respective mean of a set of positive numbers is less than or the numbers. Prove that the harmonic mean of a set of positive numbers is less than or equal to the geometric mean. When does equality occur?
- 31. Let n be a positive integer. Suppose that  $a_1, a_2, \ldots, a_n$  and  $b_1, b_2, \ldots, b_n$  are real Let n be a positive integer. Let n be a positive integer of the  $b_k$ 's is not zero. For each real number t, let  $P(t) = \sum_{k=1}^{n} \left(a_k - tb_k\right)^2.$ 
  - a) Show that P is a polynomial in t of degree 2.
  - b) By completing the square, find the value of t that minimizes P.
  - c) Show that the value of t from part (b) is the appropriate choice for the constant tthat is needed in the proof of the Cauchy-Schwarz Inequality.
- 32. Prove the Cauchy-Schwarz Inequality for the case n=2 by writing out both sides of the inequality, then multiplying and rearranging terms until a familiar inequality is obtained. Make certain that the steps are reversible!
- 33. Rephrase the Cauchy-Schwarz Inequality in the language of the vector space  $\mathbb{R}^n$ .
- **34.** Let  $a_1, a_2, \ldots, a_n$  be real numbers. Prove that  $\sum_{k=1}^n |a_k| \leq \sqrt{n} \sqrt{\sum_{k=1}^n a_k^2}$ .
- 35. Let r be a fixed positive real number. Suppose that a, b, and c are real numbers that satisfy  $a^2 + b^2 + b^2 = 2$ satisfy  $a^2 + b^2 + c^2 = r^2$ . Find the maximum value of |a| + |b| + |c| and the values of a, b, and c that generate the maximum value.
- 36. Suppose that  $a_1, a_2, \dots, a_n$  are positive real numbers. Find the minimum value of the expression  $\left(\sum_{k=1}^{n} a_k\right) \left(\sum_{k=1}^{n} \frac{1}{a_k}\right)$ .
- 37. Suppose that x > -1 and that  $x \ne 0$ . Prove that  $(1 + x)^n > 1 + nx$  for each positive integer n > 1. This result is brown. integer n > 1. This result is known as **Bernouilli's Inequality**.

# THE COMPLETENESS AXIOM

The rational numbers are closed under addition and multiplication, but the rational numbers are not closed under the process and multiplication, but the rational section is the process of the process o numbers are not closed under addition and multiplication, but the ransection 1.1 with a proof that  $\sqrt{2}$  is a finding roots. This was illustrated in are all numbers Section 1.1 with a proof that  $\sqrt{2}$  is not a rational number. The rational number rational number of sequences of the rational number of sequences of the rational number. are also not closed under the limit process since there are convergent sequences of reader has had a most converge to the sequences of the reader has had a most converge to the sequences of the reader has had a most converge to the reader had the most converge to the most c rational numbers that do not converge to a rational number. (We assume that the sequence the sequence) reader has had a little exposure to the limit process.) For example, the sequence is a sequence of rational numbers that converges to an irrational number (the limit process is central to the sequence of rational process is central to the sequence of rational numbers that converges to an irrational number (the limit process is central to the sequence of rational number (the limit process is central to the sequence of rational number (the limit process is central to the sequence of rational numbers that converges to an irrational number (the limit process is central to the sequence of rational numbers that converges to an irrational number (the limit process is central to the sequence of rational numbers that converges to an irrational number (the limit process is central to the sequence of rational numbers that converges to an irrational number (the limit process is central to the sequence of rational numbers that converges to an irrational number (the limit process is central to the sequence of rational numbers that converges to an irrational number (the limit process is central to the sequence of rational numbers that converges to an irrational number (the limit process is central to the sequence of rational numbers). is a number whose decimal expansion has no repeating pattern). Since the limit of the development of the rational number (the limit of the development of the rational number (the limit of the development of the rational number (the limit of the development of the rational number (the limit of the rational number of the rational number (the limit of the rational number of the limit of the development of the rational number (the limit of the limit of the limit of the development of the rational number of the limit of the limit of the development of the limit o

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## DEFINI

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## DEFINIT

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- **b**)

It sho a proof of called the

- 8. Prove that a nonempty set that is bounded above has only one supremum.
- 8. Prove that a nonchard.
  9. The Completeness Axiom only asserts something about sets that are bounded about sets are set of real particles. The Completeness Axiom to prove that every nonempty set of real numbers that bounded below has an infimum.
- 10. Prove that the infimum and supremum of the interval (a, b) are a and b, respectively
- 11. Prove that a nonempty finite set contains its infimum.
- 12. Let S be a nonempty set of real numbers that is bounded above and let  $\beta = 300$ Prove that for each  $\epsilon > 0$  there exists a point  $x \in S$  such that  $x > \beta - \epsilon$ .
- 13. Find the supremum of the set  $\{x: 3x^2 + 3 < 10x\}$ .
- 14. Use the Completeness Axiom to finish the proof of Theorem 1.9, that is, to prove to an interval has one of nine possible forms. There are a number of cases to consider the set is bounded, the set is bounded above but not below, etc.
- 15. Let a be a positive number. Prove that for each real number x there is an integer such that  $na \le x < (n+1)a$ .
- **16.** Referring to Theorem 1.17, prove  $(1) \Rightarrow (2)$ .
- 17. Referring to Theorem 1.17, prove  $(3) \Rightarrow (4)$ .
- 18. Prove each of the following results—give a direct proof of each one—without use the Completeness Axiom. These results could be called the Archimedean Property the rational numbers.
  - a) If a and b are positive rational numbers, then there exists a positive integer n such that na > bthat na > b.
  - b) For each positive integer n, there exists a rational number r such that r > n.
  - c) For each rational number x, there exists an integer n such that  $n \le x < n + 1$ .

  - d) For each positive rational number x, there is a positive integer n such that  $1/n^{-\zeta 1}$ Let s and t be real result. 19. Let s and t be real numbers such that t - s > 1. Prove that there exists an integral such that s .
  - 20. Let x be a real number. Prove the following statement: for each  $\epsilon > 0$ , there exists rational number r such that 0 = 1.
  - rational number r such that  $0 < |x r| < \epsilon$ . 21. A real number of the form  $p/2^n$ , where p is an integer and n is a nonnegative integrational number. pis known as a **dyadic rational number**. Prove that there is a dyadic rational number. between any two distinct real numbers.
  - 22. Consider the set A defined in the proof of Theorem 1.19. Prove that b is an analysis and the set A defined in the proof of Theorem 1.19.

  - 23. Use an argument similar to the one in the proof of Theorem 1.19 to prove a) there is a real number x such that  $x^2 = 2$ ;
  - b) there is a real number y such that  $y^3 = 7$ .

  - 24. Prove that every decimal expansion represents a real number. The rest of the exercises in this section are in no particular order. 25. Prove that each real number  $x \in [0, 1]$

- 26. Let A be that inf A
- **27.** Let *A* be that  $a \leq a$ and only
- 28. Let A and inf(
  - Is there a either a p
- **29.** Let *S* be where -
  - **30.** Let *S* be be the set related to and k < 0
  - 31. Let A an  $C = \{ab\}$ valid if ei

#### Coul 1.4

The Complete from the set of not Q (such as Axiom. This rational numb the set of ratio Since every r not rational no than the set of numbers than have no eleme that one set is

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first agree on an infinite set

### Chapter 1 Real Numbers 36

- b) Use the idea in part (a) to determine an explicit one-to-one correspondence between  $\mathbb{Z}^+$  and the set of all ordered pairs of positive integers.
- c) Use part (a) to give a different proof of Theorem 1.26.
- **15.** Prove Lemma 1.27.
- **16.** Prove Lemma 1.28.
- 17. Suppose that a and b are distinct real numbers such that a < b.
  - a) Prove that the set  $\{x \in \mathbb{R} : a < x < b\}$  is uncountable.
  - **b**) Prove that the set  $\{x \in \mathbb{Q} : a < x < b\}$  is countably infinite.

The rest of the exercises in this section are in no particular order.

18. Let A be the collection of all sequences of 0's and 1's. One example of an element the set A is the sequence

 $1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \dots$ 

Let B be the subset of A that consists of all sequences of 0's and 1's for which the number of 1's is finite. One example of an element of the set B is the sequence

 $1, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \dots$ 

- a) Use Theorem 1.25 to show that the set B is countably infinite.
- b) Prove that the set A is uncountable. (The standard proof of this result is known? Canton's diagonal Cantor's diagonal process. Assume that the set A is countably infinite, that is each positive integer. each positive integer corresponds to one sequence in A. Write out the sequence underneath each other underneath each other, a first sequence, a second sequence, and so on resulting figure looks the diagonal of t resulting figure looks like a matrix composed of 0's and 1's. Use the diagonal of this matrix to generate a composed of 0's and 1's. this matrix to generate a sequence in A that is not in the list.)
- c) Prove that the collection of all subsets of positive integers is uncountable of a sequence in A that is not in the list.)

establishing a one-to-one correspondence between this set and the set A.

Use Exercise 25. d) Use Exercise 25 in Section 1.3 and parts (a) and (b) to prove that the interval [0,1]

- 19. Use the fact that every real number has a unique decimal expansion that does not end to the state of the
- in all 9's to prove that the interval (0, 1) is an uncountable set. 20. Let \$\mathcal{F}\$ be a collection of disjoint intervals of real numbers. This means that if \$l\$ and \$l\$
  21. Prove that the intervals in \$\mathcal{F}\$, then \$l\$ O \$l\$.
- are two distinct intervals in  $\mathcal{F}$ , then  $I \cap J = \emptyset$ . Prove that every infinite set countable. 21. Prove that every infinite set contains a countably infinite subset.
- 22. Prove that a set is infinite set contains a countably infinite subset.
  23. The set of real subsets. (A set A is a point of the set of real subsets) with one of its proper subsets. (A set A is a proper subset of B if  $A \subseteq B$  but  $A \not= B$ ) bers. Another classic composed as a proper subset of B if  $A \subseteq B$  but  $A \not= B$ . 23. The set of real numbers is composed of the rational numbers and the irrational numbers. An algebraic of real number of real numbers and the irrational numbers and transcendent of real numbers. bers. Another classification of real numbers and the irrational numbers. An **algebraic number** is any numbers involves algebraic and transcendent numbers. numbers. An **algebraic number** is any number that is a root of a polynomial  $x^2 - 2$ . A transitional number that is a root of a polynomial  $x^2 - 2$ . A transitional number is any number that is a root of a polynomial  $x^2 - 2$ . integer coefficients. For example, the number  $\sqrt{2}$  is algebraic since it is a root of a polynomial  $\sqrt{2}$ . A **transcendental number**  $\sqrt{2}$  is algebraic since it is a root and algebraic since are algebraic since it is a root of a polynomial  $\sqrt{2}$ . polynomial  $x^2 - 2$ . A **transcendental** number  $\sqrt{2}$  is algebraic since it is a root and algebraic since it is a root of a polynomial  $x^2 - 2$ . A **transcendental** number is a real number that is not an algebraic since it is a root of a polynomial  $x^2 - 2$ .

  - a) prove that every rational number is an algebraic number. b) Prove that every rational number is an algebraic number.
    c) Prove that 2 \_ /5

- d) Prove that the
- e) Prove that th proof that a sp difficult. The
- 24. Although there ex sets (see Exercise A and B are unc correspondence be set and let  $\mathcal{P}(A)$  1 power set of A.
  - a) Suppose that .
  - **b**) Suppose that .
  - c) Suppose that A between A and
- 25. Let S be an uncou there is a one-to-or

#### REAL-VALU 1.5

A real-valued function output values are real in this text will be a interval. Consequently or implicitly) of the fo function f is defined o this section, we will di adjectives used to desc course has had a great material in this section a reference as the read

Let n be a positive form

P(x)

where  $a_0, a_1, \ldots, a_n$  ar as the coefficients of F every point, is referred polynomial is a real nur two polynomials. The since the constant func rational function at a pa subtraction, multiplicat involves these four oper the functions f and g d

$$f(x) = \sqrt[3]{}$$