DEFINITION 1.3 An **ordered field** is a field F that is an ordered set with the following additional properties:

1. If $x > 0$ and $y > 0$, then $x + y > 0$.
2. If $x > 0$ and $y > 0$, then $xy > 0$.
3. $x < y$ if and only if $y - x > 0$.

For the record, there are other ways to define an ordered field, and even within the approach that has been adopted here, there are variations. In fact, Definition 1.3 involves a degree of redundancy: it is possible to prove that property (3) implies property (1). However, the three properties listed in Definition 1.3 are very familiar properties: the sum of two positive numbers is positive, the product of two positive numbers is positive, and x is less than y if and only if $y - x$ is positive.

All of the familiar properties of equalities and inequalities are valid in an ordered field. As a typical example, the property,

$$\text{if } x < y \text{ and } z > 0, \text{ then } xz < yz,$$

follows from the properties of an ordered field. We will neither state nor prove these results here and simply assume that the reader is familiar with these properties of real numbers. In any case, proving these properties is not the purpose of this textbook. Once again, the interested reader is referred to a book on abstract algebra.

The two most familiar examples of ordered fields are the set of rational numbers and the set of real numbers. (As an aside, the set of complex numbers is a field that is not an ordered field.) In other words, the field properties and the order properties do not distinguish between the sets $\mathbb{Q}$ and $\mathbb{R}$; both sets satisfy all of the properties that have been listed thus far. Since the sets $\mathbb{Q}$ and $\mathbb{R}$ have some differences, the real numbers must possess some additional property that the rational numbers do not possess. A discussion of the distinction between these two sets of numbers will be the topic of Section 1.3.

Exercises

1. It was stated in the text that the rational numbers form a field. In particular, the rational numbers are closed under addition and multiplication. Prove this fact by showing that the sum of two rational numbers is a rational number and the product of two rational numbers is a rational number.
2. Prove that there is a rational number between any two distinct rational numbers.
3. Convert each of the rational numbers into a repeating decimal.
   a) $8/27$ b) $4/21$ c) $5/19$
4. Convert each repeating decimal into a rational number of the form p/q, where p and q are positive integers with no common divisors.
   a) $0.357357357\ldots$ b) $0.327272727\ldots$ c) $0.21153846153846\ldots$
5. Find the millionth digit in the decimal expansion of $2/7$.
6. Prove that the reciprocal of an irrational number is an irrational number.
7. Prove that the sum of a rational number and an irrational number is irrational.
8. Prove that the product of a nonzero rational number and an irrational number is an irrational number.

9. Let x be an irrational number. Prove that there exists an irrational number y such that xy is a rational number.
10. Let x be a real number. Prove that at least one of the numbers $\sqrt{2} - x$ or $\sqrt{2} + x$ is irrational.
11. Let n be a positive integer that is not a perfect square. Prove that $\sqrt{n}$ is irrational.
12. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.
13. Prove that $\sqrt{n-1} + \sqrt{n+1}$ is irrational for every positive integer n.
14. Prove that there is no rational number r such that $2^r = 3$.
15. Let x and y be irrational numbers such that $x - y$ is also irrational. Define sets A and B by $A = \{x + r : r \in \mathbb{Q}\}$ and $B = \{y + r : r \in \mathbb{Q}\}$. Prove that the sets A and B have no elements in common.
16. Let A be the set of all numbers of the form $a + b\sqrt{2}$, where a and b are arbitrary rational numbers. Let addition and multiplication be defined on A in the same way they are defined for real numbers. Prove that the set A is a field.
17. Let B be the set of all irrational numbers together with the numbers 0, 1, and -1. Let addition and multiplication be defined on B in the same way they are defined for real numbers. Determine the field properties that are satisfied by B. Is B a field?
18. Let S be the set of all ordered pairs of positive integers and adopt the convention that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. For each given relation, determine whether or not the relation satisfies the properties listed in Definition 1.2.
    a) Define a relation $<$ on S by $(a, b) < (c, d)$ if and only if $ab < cd$.
    b) Define a relation $<$ on S by $(a, b) < (c, d)$ if and only if $ad < bc$.
19. Referring to Definition 1.3, prove that property (3) implies property (1). Be certain to use only properties that have been proved or already assumed to be true.
20. Use the properties of an ordered field to prove the following: if $x < y$ and $z > 0$, then $xz < yz$.

1.2 Absolute Value, Intervals, and Inequalities

This section contains a discussion of the absolute value function, the concept of an interval, a formula for geometric sums, and a brief exploration of two interesting inequalities. The properties of the absolute value function will be used throughout the text, primarily as a measure of the distance between two numbers. Almost all the theorems discussed in this book are stated for functions that are defined on an interval, so it is important to understand exactly what is meant by the term "interval". The formula for a geometric sum is quite simple and appears now and again in proofs and examples. Although the last two inequalities discussed in this section will not be used in this text, they are important in other areas of real analysis and are related to ideas that have been considered thus far.

In analysis, it is often necessary to measure the distance between two points. Since the points considered in analysis may represent numbers, vectors, functions, or sets, the notion of distance between points and how to compute it can become rather abstract. It is possible to study this notion in a more general setting (see the discussion of metric spaces in Section 8.5), but at this stage of the game, we will remain in the familiar territory of the real numbers. The absolute value function can