Math 35

Winter 2014

February 25 Sample Solutions

Our formal definition of the convergence of a sequence of real numbers is:

Suppose that $\{x_n\}$ is a sequence of real numbers, and x is a real number. Then the sequence $\{x_n\}$ converges to x if and only if

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{Z}^+)(\forall n \in \mathbb{Z}^+)\big(n \geq N \implies |x_n - L| < \varepsilon\big).$$

Exercise 1: Propose a formal definition of the convergence of a sequence of functions.

Warning: Any variable you include in your definition requires a quantifier.

Suppose that $\{f_n\}$ is a sequence of functions on an interval I, and f is a function on I. Then the sequence $\{f_n\}$ converges to f if and only if

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Solution: We proposed two different definitions:

$$(\forall x \in I)(\forall \epsilon > 0)(\exists N)(\forall n \geq N)\ \big(|f_n(x) - f(x)| < \varepsilon\big);$$

$$(\forall \epsilon > 0)(\exists N)(\forall x \in I)(\forall n \geq N)\ \big(|f_n(x) - f(x)| < \varepsilon\big).$$

The first is the definition of *pointwise convergence* and the second is the definition of *uniform convergence.*

Note: If $\{x_n\}$ is a sequence of real numbers converging to x, we may write $\lim_{n\to\infty} x_n = x$. However, you will never see in our textbook the expression $\lim_{n\to\infty} f_n = f$.

This is because there are two different notions of convergence of a sequence of functions, and therefore there are two different kinds of limit. We do not want to use notation that does not specify what kind of convergence, and what kind of limit, we mean.

You may see the expression $\lim_{n\to\infty} f_n(x) = f(x)$. This is perfectly fine; it means that for the number x (whatever x may happen to be), the sequence of real numbers $\{f_n(x)\}$ converges to the real number $f(x)$. It does *not* mean that the sequence of functions $\{f_n\}$ converges to the function f.

This is one reason why the textbook, on page 242, is so emphatic about the distinction between "f" and "$f(x)$."

Exercise 2: Apply your definition of convergence to the sequence $\{f_n\}$ of functions on $[-1,1]$ defined by $f_n(x) = \sqrt[2n+1]{x}$. Does this sequence converge? If so, to what function?

Solution: This sequence of functions converges pointwise to the function defined by

$$f(x) = \begin{cases} -1 & \text{if } x < 0; \\ 0 & \text{if } x = 0; \\ 1 & \text{if } x > 0. \end{cases}$$

It does not converge uniformly.

Exercise 3: For the sequence of functions in Exercise 2, determine

$$\lim_{n\to\infty} \lim_{x\to 0} f_n(x);$$

$$\lim_{x\to 0} \lim_{n\to\infty} f_n(x).$$

Solution:

$$\lim_{n\to\infty} \lim_{x\to 0} f_n(x) = \lim_{n\to\infty} 0 = 0;$$
$$\lim_{x\to 0} \lim_{n\to\infty} f_n(x) = \lim_{x\to 0} f(x), \text{ which is not defined.}$$

Exercise 4: Apply any other proposed definitions of convergence to the sequence in Exercise 2.

Solution: These were our only proposed definitions. You might now want to think about this sequence of functions restricted to the interval $[0,1]$, if convergence is defined as follows.

Suppose that $\{f_n\}$ is a sequence of continuous functions on the interval $I = [0,1]$, and f is a continuous function on I. Then the sequence $\{f_n\}$ is said to $||\ ||$-converge to f if and only if

$$(\forall \epsilon > 0)(\exists N)(\forall n \geq N)\ \big(||f_n - f|| < \varepsilon\big),$$

where we define the norm of f (intuitively, a measure of the size of f; this is not the only possible definition of norm) by

$$||f|| = \sqrt{\int_0^1 (f(t))^2\, dt}.$$

Hint: Considering the function to which $\{f_n\}$ converges pointwise, what continuous function would be the most likely candidate for this sequence to $||\ ||$-converge to? Does it?

Exercise 5: Consider the possible convergence (in any proposed sense) of the sequence of functions on $(-1, 1)$ defined by $f_n(x) = \sum_{k=0}^{n} x^k$.

Solution: This sequence of functions converges pointwise to f defined by $f(x) = \frac{1}{1-x}$. It does not converge uniformly.

You can think about $|| \, ||$-convergence.

Exercise 6: For the sequence of functions in Exercise 5, consider

$$\lim_{n\to\infty} \lim_{x\to -1^+} f_n(x);$$

$$\lim_{x\to -1^+} \lim_{n\to\infty} f_n(x).$$

Solution: Assuming our sequence begins with $n = 0$, define

$$s_n = \sum_{k=0}^{n} (-1)^k = \begin{cases} 1 & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Then

$$\lim_{n\to\infty} \lim_{x\to -1^+} f_n(x) = \lim_{n\to\infty} s_n, \text{ which is not defined.;}$$

$$\lim_{x\to -1^+} \lim_{n\to\infty} f_n(x) = \lim_{x\to -1^+} \frac{1}{1-x} = \frac{1}{2}.$$