Math 35
Winter 2014
January 28

Work in a group of two or three people.

Exercise 1 The *greatest integer* function is defined in the textbook. The definition is:

$\lfloor x \rfloor$ is the greatest integer n such that $n \leq x$.

Intuitively, $\lfloor x \rfloor$ is the whole number part of x (at least for positive x). For example $\lfloor 1\frac{1}{2} \rfloor = 1$, $\lfloor 2 \rfloor = 2$, $\lfloor \pi \rfloor = 3$, and $\lfloor -1.5 \rfloor = -2$.

Let z be any real number. Find the limit L of the sequence

$$\left\{ \frac{\lfloor nz \rfloor}{n} \right\}$$

(You need not prove anything yet.)

Solution: For any x, by the definition of $\lfloor x \rfloor$, we have $\lfloor x \rfloor \leq x$ and $\lfloor x \rfloor + 1 > x$. Subtracting 1 from both sides of the second inequality gives $\lfloor x \rfloor > x - 1$. Putting these together,

$$x - 1 < \lfloor x \rfloor \leq x.$$

Substituting nz for x gives

$$nz - 1 < \lfloor nz \rfloor \leq nz.$$

Dividing by n gives

$$z - \frac{1}{n} < \frac{\lfloor nz \rfloor}{n} \leq z.$$

From this we see that $L = z$.

Exercise 2: Prove the sequence of exercise 1 converges to the limit L in two different ways:

1. directly from the definition of convergence;

2. using the theorems stated in Chapter 2 of the textbook.

Proposition: For any real number z, the sequence $\left\{\frac{\lfloor nz \rfloor}{n}\right\}$ converges to z.

Proof 1: For any x, by the definition of $\lfloor x \rfloor$, we have $\lfloor x \rfloor \leq x$ and $\lfloor x \rfloor + 1 > x$. Subtracting 1 from both sides of the second inequality gives $\lfloor x \rfloor > x - 1$. Putting these together,

$$x - 1 < \lfloor x \rfloor \leq x.$$

Substituting nz for x gives

$$nz - 1 < \lfloor nz \rfloor \leq nz.$$

Dividing by n gives

$$z - \frac{1}{n} < \frac{\lfloor nz \rfloor}{n} \leq z.$$

From this we see that $\left|\frac{\lfloor nz \rfloor}{n} - z\right| < \frac{1}{n}$.

Now, let $\varepsilon > 0$ be given, and choose N such that $\frac{1}{N} < \varepsilon$. Then, for any $n \geq N$, we have

$$\left|\frac{\lfloor nz \rfloor}{n} - z\right| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

This proves that $\left\{\frac{\lfloor nz \rfloor}{n}\right\}$ converges to z. $\square$

Proof 2: For any x, by the definition of $\lfloor x \rfloor$, we have $\lfloor x \rfloor \leq x$ and $\lfloor x \rfloor + 1 > x$. Subtracting 1 from both sides of the second inequality gives $\lfloor x \rfloor > x - 1$. Putting these together,

$$x - 1 < \lfloor x \rfloor \leq x.$$

Substituting nz for x gives

$$nz - 1 < \lfloor nz \rfloor \leq nz.$$

Dividing by n gives

$$z - \frac{1}{n} < \frac{\lfloor nz \rfloor}{n} \leq z.$$

The constant sequence $\{z\}$ converges to z. The sequence $\left\{\frac{1}{n}\right\}$ converges to 0, so by Theorem 2.7 the sequence $\left\{z - \frac{1}{n}\right\}$ converges to $z - 0$; that is, to z. Therefore, by the squeeze theorem, $\left\{\frac{\lfloor nz \rfloor}{n}\right\}$ also converges to z. $\square$

Exercise 3: Suppose that $\{x_n\}$ is a sequence, S is the set of subsequential limits of $\{x_n\}$, and $a \in S$. Prove that for every $\varepsilon > 0$ and every N, there is an $n > N$ such that $|a - x_n| < \varepsilon$. (This is really a lemma for the next exercise.)

Lemma: Suppose that $\{x_n\}$ is a sequence, S is the set of subsequential limits of $\{x_n\}$, and $a \in S$. For every $\varepsilon > 0$ and every N, there is an $n > N$ such that $|a - x_n| < \varepsilon$.

Proof: Let $\{x_{p(m)}\}$ be a subsequence converging to a. Choose M such that, for $m \geq M$, we have $|a - x_{p(m)}| < \varepsilon$. (This is possible by the definition of convergence.) Now choose m large enough so that $p(m) > \max\{M, N\}$. (This is possible because $\{p(m)\}$ is an increasing sequence.) Set $n = p(m)$. □

Exercise 4: Suppose that $\{x_n\}$ is a sequence, S is the set of subsequential limits of $\{x_n\}$, and $\{a_n\}$ is a sequence of elements of S that converges to a. Prove that $a \in S$.

Proposition: Suppose that $\{x_n\}$ is a sequence, S is the set of subsequential limits of $\{x_n\}$, and $\{a_n\}$ is a sequence of elements of S that converges to a. Then $a \in S$.

Proof: Inductively define a subsequence $\{x_{p(n)}\}$ converging to a.

To guarantee $\{x_{p(n)}\}$ converges to a, it is enough to guarantee that, for all n, we have $\left|a - x_{p(n+1)}\right| < \frac{2}{n}$. (See the footnote.[1])

Set $p(1) = 1$.

Suppose that $p(n)$ has been defined. We must choose $p(n+1) > p(n)$ such that $\left|a - x_{p(n+1)}\right| < \frac{2}{n}$.

Choose m such that $|a - a_m| < \frac{1}{n}$. This is possible because the sequence $\{a_n\}$ converges to a.

Now choose $k > p(n)$ such that $|a_m - x_k| < \frac{1}{n}$. This is possible by the lemma, with $N = p(n)$, $\varepsilon = \frac{1}{n}$, and a_m in place of a.

By the triangle inequality, $|a - x_k| \leq |a - a_m| + |a_m - x_k| < \frac{2}{n}$.

Set $p(n+1) = k$. Then we have $p(n) = k < p(n+1)$ and $\left|a - x_{p(n+1)}\right| = |a - x_k| < \frac{2}{n}$, as required. □

[1]There is nothing magic about $\frac{2}{n}$. Anything that approaches 0 as n approaches ∞ would suffice.

Exercise 5: Determine whether the sequences in part (a) converge. Find the limits of the sequences in part (b). Theorem 2.14 of the textbook may be useful.

Note: For these problems, I haven't always cited all the relevant theorems.

(a.) $\{\sqrt{n+1}-\sqrt{n}\}$ $\left\{\frac{n!}{n^n}\right\}$ $\left\{\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\right\}$

For the first sequence, we do a little algebra:

$$\sqrt{n+1}-\sqrt{n} = \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})} = \frac{1}{\sqrt{n+1}+\sqrt{n}},$$

which converges to 0, because the denominator diverges to $+\infty$.

For the second sequence, we set $a_n = \frac{n!}{n^n}$, and derive a recurrence relation:

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}} = \frac{n!(n+1)}{(n+1)^n(n+1)} = \frac{n!(n^n)}{(n+1)^n(n^n)} = \frac{n^n}{(n+1)^n}\,\frac{n!}{n^n} = \left(\frac{n}{n+1}\right)^n a_n.$$

In particular, since all the terms a_n are positive and $\frac{n}{n+1} < 1$, this shows $a_{n+1} < a_n$. That is, we have a decreasing sequence of positive numbers, which therefore converges.

For the third sequence, we set $a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}$. We can see that all terms are positive, and that $a_{n+1} = \frac{2n+1}{n+1}\,a_n$. Now, the sequence $\left\{\frac{2n+1}{n+1}\right\}$ converges to 2, so it is eventually always greater than $\frac{3}{2}$. That is, for some N, for all $n \geq N$ we have $a_{n+1} > \frac{3}{2}\,a_n$. From this we can show that, for $m \geq 1$, we have $a_{N+m} > \left(\frac{3}{2}\right)^m a_N$. Since by Theorem 2.14 the sequence $\left\{\left(\frac{3}{2}\right)^m\right\}$ diverges to ∞, so does our sequence.

(b.) $\{(\sqrt[n]{n}-1)^n\}$ $\left\{\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}\right\}$ $\left\{\frac{2^n+n^2}{5^n-n}\right\}$

For the first sequence, we can see the terms are positive. By Theorem 2.14 we know that $\{\sqrt[n]{n}\}$ converges to 1, so $\{\sqrt[n]{n}-1\}$ converges to 0, so for large enough n we have $0 < \sqrt[n]{n}-1 < \frac{1}{2}$, and $0 < (\sqrt[n]{n}-1)^n < \frac{1}{2^n}$. Since (by the same theorem) the sequence $\left\{\frac{1}{2^n}\right\}$ converges to 0, so, by the squeeze theorem, does our sequence.

For the second sequence, consider a fixed n.

For $1 \le k \le n$, we have $n < \sqrt{n^2+k} \le \sqrt{n^2+n}$, so $\frac{1}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+k}} < \frac{1}{n}$, and

$$\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+n}} \le \sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}} \le \sum_{k=1}^{n}\frac{1}{n}.$$

The terms in the lefthand and righthand sums do not depend on k, so we have

$$\frac{n}{\sqrt{n^2+n}} \le \sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}} \le \frac{n}{n}.$$

The righthand term equals 1, and the lefthand term equals $\sqrt{\frac{n}{n+1}}$, which converges to 1 as n approaches ∞. Therefore, our sequence converges to 1.

For the third sequence, we can rewrite

$$\frac{2^n+n^2}{5^n-n} = \frac{\left(\frac{2}{5}\right)^n - n^2 5^{-n}}{1-n5^{-n}}.$$

Since $\left\{\left(\frac{2}{5}\right)^n\right\}$ converges to 0, to show our sequence converges to 0 as well, we need only show that the sequences $\{n5^{-n}\}$ and $\{n^2 5^{-n}\}$ converge to 0.

We can do this by a method similar to the one we used for the third sequence of part (a): Let $a_n = n^2 5^{-n}$. Then $a_{n+1} = \frac{1}{5}\left(\frac{n+1}{n}\right)^2 a_n$. Since the sequence $\left\{\frac{1}{5}\left(\frac{n+1}{n}\right)^2\right\}$ converges to $\frac{1}{5}$, it is eventually less than $\frac{1}{2}$. Then, by an argument like our previous one, there is some N such that $a_{N+m} \le \left(\frac{1}{2}\right)^m a_N$ for all $m \ge 1$, and since the sequence $\left\{\left(\frac{1}{2}\right)^m\right\}$ converges to 0, so does $\{a_n\}$.

To prove $\{a_n\}$ converges, directly from the definition of convergence:

Proposition: The sequence $\{a_n\}$ converges to L.
[You get to choose L. Of course, you choose L to be the limit of the sequence.]

Proof: Let $\varepsilon > 0$.
[You don't get to choose ε. Your argument must work for every $\varepsilon > 0$.]

Define N to be ________________________.
[You get to define N. Generally, the definition of N will depend on ε.]

Let $n \geq N$.
[You don't get to choose n. Your argument must work for every $n \geq N$.]

We will prove $|a_n - L| < \varepsilon$.
[Now you have to prove this.]

To prove $\{a_n\}$ diverges, directly from the definition of divergence:

Proposition: The sequence $\{a_n\}$ diverges.

Proof: Let L be a real number.
[You don't get to choose L. Your argument must work for every L. Division into cases is common here.]

Define ε to be ________________________.
[You get to define ε; you must have $\varepsilon > 0$. Generally, the definition of ε will depend on L.]

Let N be any number
[You don't get to choose N. Your argument must work for every N.]

Define n to be ________________________.
[You get to define n; you must have $n \geq N$. Generally, the definition of n will depend on N and L.]

We will prove $|a_n - L| \geq \varepsilon$.
[Now you have to prove this.]

[These last two steps can be combined: "We will prove there is an $n \geq N$ such that $|a_n - L| \geq \varepsilon$." Doing this allows you to use, for example, proof by contradiction, without trying to figure out exactly what n should be.]