**Math 35**
**Winter 2014**
**Some Sample Solutions from Wednesday, January 22**

**Exercise 1:** Give a careful proof that a bounded increasing sequence $\{a_n\}$ converges to $a = \sup\{a_n \mid n \in \mathbb{Z}^+\}$.

**Proof:** Let $\{a_n\}$ be a bounded increasing sequence. Since $\{a_n\}$ is bounded, $a = \sup\{a_n \mid n \in \mathbb{Z}^+\}$ exists by the completeness axiom. We must show $\{a_n\}$ converges to $a$.

Let $\varepsilon > 0$.

Since $a$ is the least upper bound of $\{a_n \mid n \in \mathbb{Z}^+\}$, then $a - \varepsilon$ is not an upper bound. Therefore there is some $N$ such that $a_N > a - \varepsilon$.

Choose any such $N$. We will show that, for all $n \geq N$, we have $|a - a_n| < \varepsilon$. This will complete the proof that $\{a_n\}$ converges to $a$.

To do this, suppose $n \geq N$. By our choice of $N$, we have $a - \varepsilon < a_N$. Because the sequence is increasing, we have $a_N \leq a_n$. Because $a$ is an upper bound for the terms of the sequence, we have $a_n \leq a$.

Since $a - \varepsilon < a_N \leq a_n \leq a$, we have $|a - a_n| < \varepsilon$, which is what we needed to show.

**Exercise 2:** Define a sequence $\{x_n\}$ to be *eventually bounded above* by $a$ if there is $N$ such that

$$(\forall n \geq N)[x_n \leq a],$$

and *eventually bounded below* by $b$ if there is $M$ such that

$$(\forall n \geq M)[x_n \geq b].$$

(c.) Complete the following, using the concepts "eventually bounded above" and "eventually bounded below":

The sequence $\{x_n\}$ converges to the real number $L$ if and only if

**Claim:** The sequence $\{x_n\}$ converges to the real number $L$ if and only if, for every $\varepsilon > 0$, the sequence $\{x_n\}$ is eventually bounded above by $L + \varepsilon$ and eventually bounded below by $L - \varepsilon$.

**Exercise 3:** A sequence $\{a_n\}$ is called a *Cauchy sequence* if it satisfies

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{Z}^+)(\forall m, n \geq N)[|a_m - a_n| < \varepsilon].$$

This says that by going far enough out in the sequence, you can guarantee that $a_m$ and $a_n$ are close to each other, regardless of how far apart $m$ and $n$ are.

(a.) Show that the sequence $\{a_n\}$ given by

$$a_n = \sum_{k=0}^{n} 3(.24)^k$$

is a Cauchy sequence.

Let $r = .24$. Then we have

$$a_n = \sum_{k=0}^{n} 3r^k$$

and, for $m > n$,

$$0 < a_m - a_n = \sum_{k=n+1}^{m} 3r^k = 3r^n \sum_{i=1}^{m-n} r^k = 3r^n \frac{1 - r^{m-n+1}}{1-r} < 3r^n \frac{1}{1-r} = \frac{3}{.76} r^n.$$

We will use the fact that since $0 < r < 1$, the sequence $\{r^n\}$ converges to 0 and is decreasing.

To show $\{a_n\}$ is a Cauchy sequence, let $\varepsilon > 0$, and choose $N$ such that $\frac{3}{.76} r^N < \varepsilon$. (We can do this because $\{r^n\}$ converges to 0.)

Now, suppose $m, n \geq N$, and without loss of generality say $m > n$. Using the fact that for $N \leq n$ we have $r^N \geq r^n$, we can write

$$0 < a_m - a_n \leq \frac{3}{.76} r^n \leq \frac{3}{.76} r^N < \varepsilon.$$

Therefore $|a_n - a_n| < \varepsilon$ for all $n, m < N$, which is what we needed to show.

**Exercise 4:** Give an example of a sequence $\{a_n\}$ that is *not* a Cauchy sequence, but that does satisfy

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{Z}^+)(\forall n \geq N)[|a_{n+1} - a_n| < \varepsilon].$$

This says merely that by going far enough out in the sequence, you can guarantee that $a_n$ is close to $a_{n+1}$.

For this problem in particular, you might want to describe your sequence by writing out enough terms to make a pattern apparent, rather than trying to find a formula for $a_n$. You need not prove anything. If it is not obvious that your sequence works, you should explain informally why it does.

Hint: You can go a long distance in tiny steps by taking many steps.

**Possible Solution:**

$(0,\ 1,\ 1\frac{1}{2},\ 2,\ 2\frac{1}{3},\ 2\frac{2}{3},\ 3,\ 3\frac{1}{4},\ 3\frac{1}{2},\ 3\frac{3}{4},\ 4,\ 4\frac{1}{5},\ \dots)$

I think it's obvious that this works, but here's a brief, informal explanation, anyway:

This sequence approaches $+\infty$, so no matter how large $N$ is, we can find $n, m \geq N$ such that $a_n$ and $a_m$ are far apart. However, for every natural number $m$, there is an $N$ such that $a_N = m$, and for all $n \geq N$ we have $|a_{n+1} - a_n| < \frac{1}{m}$.

**Exercise 6:** Suppose $\{x_n\}$ is a Cauchy sequence.

(a.) Show $\{x_n\}$ is bounded.

This is proven in the textbook.

(b.) Given that $\{x_n\}$ is a Cauchy sequence, what can you say about the sets

$$A = \{a \mid \{x_n\} \text{ is eventually bounded above by } a\}$$

$$B = \{b \mid \{x_n\} \text{ is eventually bounded below by } b\}?$$

We will argue somewhat intuitively, using the idea that "$\{x_n\}$ is eventually bounded above by $a$" means that, if $n$ is large enough, we must have $x_n \leq a$, and similarly for "eventually bounded below".

Since $\{x_n\}$ is bounded, it is bounded both above and below, so $A \neq \emptyset$ and $B \neq \emptyset$.

It is not hard to see that if $a$ is in $A$, so is every number $y > a$, and if $b$ is in $B$, so is every number $y < b$.

Furthermore, if $a \in A$ and $b \in B$, then $b \leq a$. (For large enough $n$, we must have both $b \leq x_n$, since $\{x_n\}$ is eventually bounded below by $b$, and $x_n \leq a$, since $\{x_n\}$ is eventually bounded above by $a$.)

That is, every element of $A$ is an upper bound for $B$, and every element of $B$ is a lower bound for $A$. Therefore, $A$ is bounded below, and has a greatest lower bound $\overline{a}$, and $B$ is bounded above, and has a least upper bound $\overline{b}$.

We must have $\overline{b} \leq \overline{a}$.

We can show that, in fact, $\overline{b} = \overline{a}$. (Here is the first time we use the fact that $\{x_n\}$ is a Cauchy sequence; everything we have said up until now applies to any bounded sequence.)

Suppose not, and we have $\overline{a} - \overline{b} = h > 0$. Using the fact that $\{x_n\}$ is a Cauchy sequence, we can find $N$ such that, for all $n, m \geq N$, we have $|x_n - x_m| < \frac{h}{3}$. In particular, for all $n \geq N$ (setting $m = N$), we have $x_N - \frac{h}{3} < x_n < x_N + \frac{h}{3}$.

This means that $\{x_n\}$ is eventually bounded below by $x_N - \frac{h}{3}$, and so $x_N - \frac{h}{3} \in B$. Similarly $x_N + \frac{h}{3} \in A$. Therefore, since $\overline{b}$ is an upper bound for $B$ and $\overline{a}$ is a lower bound for $A$, we have $x_N - \frac{h}{3} \leq \overline{b} \leq \overline{a} \leq x_N + \frac{h}{3}$. But then $\overline{a}$ and $\overline{b}$ both lie in an interval of length $\frac{2h}{3}$, contradicting our assumption that their difference was $h$.

This proves $\overline{a} = \overline{b}$.

Set $L = \overline{a} = \overline{b}$. We can now prove that $\{x_n\}$ converges to $L$. By problem (2)(c) above, it is enough to show that, for every $\varepsilon > 0$, the sequence is eventually bounded above by $L + \varepsilon$ and eventually bounded below by $L - \varepsilon$. That is, we must show that $L + \varepsilon \in A$, and $L - \varepsilon \in B$.

To show that $L + \varepsilon \in A$, we use the fact that $L = \overline{a}$ is the greatest lower bound for $A$. Therefore, $L + \varepsilon$ is not a lower bound for $A$, and there is an $a \in A$ such that $a \leq L + \varepsilon$. But then also $L + \varepsilon \in A$.

The proof that $L - \varepsilon \in B$ is similar.