Math 35
Winter 2014
Monday, January 13
Notes on Exercise 1

For these exercises, you may use everything you know about the properties of $+$, $\cdot$, and $<$. In particular, you may use the fact that if x and y are positive, then $x < y$ implies $x^2 < y^2$.

Exercise 1: Prove directly from the Completeness Axiom that the set of positive integers is not bounded above.

"Directly from the Completeness Axiom" means that you may use the Completeness Axiom, but you may not use the things that were proven from the Completeness Axiom, such as the Archimedean property of the real numbers. (If we were allowed to use the Archimedean property, this would be easier.)

As we said in class, it is important to remember that a set X can have a supremum that is NOT an element of X.

The Completeness Axiom says that if a nonempty set of real numbers is bounded above, then it has a supremum. Applying this to our case, it says that if $\mathbb{Z}^+$ is bounded above, then $\mathbb{Z}^+$ has a supremum. We want to prove that $\mathbb{Z}^+$ is not bounded above. There are two natural ways to proceed.

On the one hand, we could try a proof by contradiction: Assume $\mathbb{Z}^+$ is bounded above. By the Completeness Axiom, then, $\mathbb{Z}^+$ has a supremum r. (I call it r for "real number" and not n for "natural number," because, as noted above, r is not necessarily an element of $\mathbb{Z}^+$.) Try to derive a contradiction from "r is the supremum of $\mathbb{Z}^+$."

On the other hand, we could try using the contrapositive of the Completeness Axiom. (The contrapositive of "if A then B" is "if not B then not A." A statement and its contrapositive are logically equivalent.) That is, we could show that $\mathbb{Z}^+$ has no supremum, and conclude from (the contrapositive of) the Completeness Axiom that it has no upper bound. To show $\mathbb{Z}^+$ has no supremum, we might try proof by contradiction: Assume $\mathbb{Z}^+$ does have a supremum, and try to derive a contradiction.

Both directions lead to the same conclusion: Let's try assuming that $\mathbb{Z}^+$ has a supremum (which we may call r), and deducing a contradiction.

We will probably have to use the defining property of the natural numbers, namely, that they are closed under adding 1.

Intuitively, if r is the *least* upper bound of the natural numbers, then there are natural numbers really close to r. But if n is really close to r (like within a distance of $\frac{1}{2}$), then $n + 1$ ought to be bigger than r. That would mean r isn't an upper bound after all.

Can we prove there is a natural number n within a distance $\frac{1}{2}$ of r, using the fact that r is the least upper bound of $\mathbb{Z}^+$? Well, if r is the least upper bound of $\mathbb{Z}^+$, then $r - \frac{1}{2}$ is not an upper bound of $\mathbb{Z}^+$. That is, not every natural number is less than or equal to $r - \frac{1}{2}$. That should help.

Here's another way to look at this: We want to show that $\mathbb{Z}^+$ has no least upper bound. A way to show a set has no least upper bound is to show that for every upper bound, there is a smaller upper bound. In our case, we let r be an upper bound for $\mathbb{Z}^+$, and show that $r - \frac{1}{2}$ is also an upper bound. (And we do *that* by showing that if it isn't, we get a contradiction.)