Math 35
Winter 2014
Solution to Problem from Wednesday, January 8

Exercise: Prove that if $0 < a$, $0 < b$, $0 < c$, and $c^2 = ab$, then $c \leq \dfrac{a+b}{2}$.

We will use the following facts; some are axioms, and some have been proven as exercises:

1. The sum of positive numbers is positive.
2. The product of positive numbers is positive.
3. The multiplicative inverse of a positive number is positive.
4. The number 1 is positive.
5. We can add any number to both sides of an inequality.
6. We can multiply both sides of an inequality by any positive number

Lemma 1: If x and y are positive numbers, and $x < y$, then $x^2 < y^2$.

Proof of Lemma 1: Multiplying both sides of the inequality $x < y$ by the positive number x gives $x^2 < xy$. Multiplying both sides of the inequality $x < y$ by the positive number y gives $xy < y^2$. Because $<$ is transitive, it follows that $x^2 < y^2$.

Lemma 2: If x and y are positive numbers, and $x^2 < y^2$, then $x < y$.

Proof of Lemma 2: Suppose $x^2 < y^2$ but $x \geq y$, and deduce a contradiction. Since $x \geq y$, there are two cases. Either $x = y$, in which case we immediately have $x^2 = y^2$, or $y < x$, in which case we have $y^2 < x^2$ by Lemma 1. In either case, we have a contradiction to the hypothesis $x^2 < y^2$. Therefore, we cannot have $x \geq y$, and so we must have $x < y$.

Proposition: If $0 < a$, $0 < b$, $0 < c$, and $c^2 = ab$, then $c \leq \dfrac{a+b}{2}$.

Proof of Proposition: Suppose that a, b, and c are positive numbers such that $c^2 = ab$. We must show that $c \leq \dfrac{a+b}{2}$.

Because 1 is positive, we know that $\dfrac{1}{2} = (1+1)^{-1}$ is positive. Therefore, because a and b are also positive, we know that $\dfrac{a+b}{2} = (a+b) \cdot \dfrac{1}{2}$ is positive as well.

By Lemma 2, then, it will be enough to show that $c^2 \leq \left(\dfrac{a+b}{2}\right)^2$. Since $c^2 = ab$, it is enough to show that

$$ab \leq \left(\frac{a+b}{2}\right)^2.$$

We will prove this by cases:

Case 1: $a \leq b$. Adding $-a$ to both sides, we have $0 \leq b - a$. Writing $h = b - a$, so $b = a + h$, we have

$$ab = a(a+h) = a^2 + ah$$

$$\left(\frac{a+b}{2}\right)^2 = \left(\frac{a+a+h}{2}\right)^2 = \left(a + \frac{h}{2}\right)^2 = a^2 + ah + \left(\frac{h}{2}\right)^2.$$

Substituting, then, we need to show that

$$a^2 + ah \leq a^2 + ah + \left(\frac{h}{2}\right)^2.$$

To see this is true: Because $h = b - a$ is positive or zero, so is $\left(\dfrac{h}{2}\right)^2 = h \cdot h \cdot \dfrac{1}{2}$. That is, we have

$$0 \leq \left(\frac{h}{2}\right)^2.$$

Adding $a^2 + ah$ to both sides of this inequality gives

$$a^2 + ah < a^2 + ah + \left(\frac{h}{2}\right)^2.$$

This is what we needed to show.

Case 2: $b \leq a$. The proof in this case is exactly the same.

This shows that for any positive a, b, and c such that $c^2 = ab$, we have $c \leq \dfrac{a+b}{2}$. $\square$

Some Notes:

1. Theorems, propositions, lemmas and corollaries are all things we prove. Often the word theorem is used for major results and proposition for minor results. A lemma is a proposition that we prove because we want to use it to prove something else. A corollary to a theorem is a proposition that we can prove easily from the theorem. Which word to use is a judgment call. Generally, "proposition" is always correct.

2. You may wonder how I came up with these lemmas. I started by noticing that our hypothesis expresses c^2 (rather than c) in terms of a and b, so it seems like it would be easier to prove something about c^2. Then I wondered whether I could prove $c^2 \leq \left(\dfrac{a+b}{2}\right)^2$. When I figured out that I could, then I saw that I needed to prove Lemma 2 to finish the proof.

   In trying to prove Lemma 2, I saw that the converse (Lemma 1) was actually easier to prove. Then, I saw that Lemma 1 could be used to prove Lemma 2.

3. You may also wonder how I thought of writing $b = a + h$. I tried manipulating the inequality I was trying to prove, $c^2 \leq \left(\dfrac{a+b}{2}\right)^2$, in all kinds of ways, hoping to get something obvious. One of the things that's often worth trying is expressing one of the variables in an equation in terms of the others. The way I did that, in this case, was by writing $b = a + h$.

   The fact that I thought of doing this may also be related to the fact, which I noticed, that if $a = b$ then the inequality becomes an equality. This indicated that the difference between a and b could be important.