Math 35
Winter 2014
Monday, February 17
Sample Solutions

Recall some definitions and results from the textbook:

A partition of an interval $[c, d]$ is a finite set of points $\{x_i \mid 0 \leq i \leq n\}$ such that $c = x_0 < x_1 < \cdots < x_{n-1} < x_n = d$.

If $f : [a, b] \to \mathbb{R}$ and $[c, d] \subseteq [a, b]$, then the variation of f on $[c, d]$ is

$$V(f, [c, d]) = sup \left\{ \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \right\},$$

where the supremum is taken over all partitions of $[c, d]$. $V(f, [c, d])$ may be a non-negative real number, or $+\infty$.

The function f is of bounded variation on $[c, d]$ if $V(f, [c, d])$ is finite.

Functions of bounded variation (on $[c, d]$) are, in particular, bounded, and sums, products, and sometimes quotients of functions of bounded variation are themselves of bounded variation. (The additional condition for $\frac{f}{g}$ to be of bounded variation, provided f and g are, is that $\frac{1}{g}$ is bounded.)

If $a < c < b$, then

$$V(f, [a, b]) = V(f, [a, c]) + V(f, [c, b]).$$

Exercise 1: Show that if f is monotone on $[a, b]$, then f is of bounded variation on $[a, b]$, and in fact $V(f, [a, b]) = |f(b) - f(a)|$. (Note that f need not be continuous.)

Suppose that f is increasing on $[a, b]$. (The argument for decreasing f is similar.) Then for any partition of $[a, b]$, and any i, we have $f(x_i) \geq f(x_{i-1})$, and so $|f(x_i) - f(x_{i-1})| = f(x_i) - f(x_{i-1}) = -f(x_{i-1}) + f(x_i)$, and so

$$\begin{aligned}\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| &= \sum_{i=1}^{n} (-f(x_{i-1}) + f(x_i)) \\ &= -f(x_0) + f(x_1) - f(x_1) + f(x_2) - \cdots - f(x_{n-1}) + f(x_n) \\ &= f(x_n) - f(x_0) \\ &= f(b) - f(a).\end{aligned}$$

Therefore,

$$V(f,[a,b]) = sup\left\{\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|\right\} = f(b) - f(a).$$

Exercise 2: Find $V(f,[0,2n\pi])$, if $f(x) = \sin x$.

We will use Exercise 1 and the fact cited on page 1,

(*) If $a < c < b$, then

$$V(f,[a,b]) = V(f,[a,c]) + V(f,[c,b]).$$

First, for simplicity, we consider the case $n = 1$. We break the interval $[0,2\pi]$ up into three intervals on which $f(x)$ is monotone, $I_1 = [0,\frac{\pi}{2}]$, $I_2 = [\frac{\pi}{2},\frac{3\pi}{2}]$, $I_3 = \frac{3\pi}{2}, 2\pi]$. By Exercise 1,

$$V[f,I_1] = \left|f\left(\frac{\pi}{2}\right) - f(0)\right| = 1;$$

$$V[f,I_2] = \left|f\left(\frac{3\pi}{2}\right) - f\left(\frac{\pi}{2}\right)\right| = 2;$$

$$V[f,I_1] = \left|f(2\pi) - f\left(\frac{3\pi}{2}\right)\right| = 1.$$

Therefore, by (*),

$$V(f,[0,2\pi]) = 1 + 2 + 1 = 4.$$

Applying this same strategy to the interval $[0,2n\pi]$, we get

$$V(f,[0,2n\pi]) = 4n.$$

Exercise 3: Show that the function f defined by

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

is not of bounded variation on $[0, 1]$.

Solution: For any n, we can consider the partition $x_0 = 0$, $x_i = \frac{2}{(2n-i+2)\pi}$ for $0 < i \leq 2n$, $x_{2n+1} = 1$. For $0 < i \leq n$, we have $f(x_i) = \sin \frac{(2n-i+2)\pi}{2}$, so $f(x_i) = 0$ if i is even ($f(x_0) = 0$ as well) and $f(x_i) = \pm 1$ if i is odd. In any case, for $1 \leq i \leq 2n$ we have $|f(x_i) - f(x_{i-1})| = 1$, and so

$$\sum_{i=1}^{2n+1} |f(x_i) - f(x_{i-1})| \geq \sum_{i=1}^{2n} |f(x_i) - f(x_{i-1})| = 2n.$$

This shows that $V(f, [0, 1]) \geq 2n$. Since this holds for every n, this shows that f is not of bounded variation on $[0, 1]$.

Exercise 4: Show that for any interval $[a, b]$, any $\varepsilon > 0$, and any $M > 0$, there is a continuous function f on $[a, b]$ such that:

1. $f(a) = f(b) = 0$.
2. $(\forall x \in [a, b])(|f(x)| < \varepsilon)$.
3. $V(f, [ab]) \geq M$.

Solution: One solution is, choosing n such that $\frac{1}{n} < \varepsilon$,

$$f(x) = \frac{1}{n} \sin \left(Mn \left(\frac{(x-a)\pi}{b-a} \right) \right).$$

To see this works, note that as x goes from a to b, $x - a$ goes from 0 to $b - a$, $\dfrac{(x-a)\pi}{b-a}$ goes from 0 to π, and $Mn\left(\dfrac{(x-a)\pi}{b-a}\right)$ goes from 0 to $Mn\pi$; therefore, $f(x)$ goes from 0 to $\pm\frac{1}{n}$ and back Mn times, making the variation $V(f, [0, 1])$ equal to $2Mn\left(\frac{1}{n}\right) = 2M$.

Exercise 5: Show there is a function f that is continuous on $[0, 1]$ but is not of bounded variation on $[0, 1]$.

Solution: For each n, let f_n be a function on the interval $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ with variation $V\left(f_n, \left[\frac{1}{n+1}, \frac{1}{n}\right]\right) = n$, $|f_n(x)| \leq \frac{1}{n}$, and $f\left(\frac{1}{n+1}\right) = f\left(\frac{1}{n}\right) = 0$. (An example of such a function is given in Exercise 4.) Then define f on $[0,1]$ by

$$f(x) = \begin{cases} f_n(x) & \text{if } \frac{1}{n+1} < x \leq \frac{1}{n}; \\ 0 & \text{if } x = 0. \end{cases}$$