m33s06: Homework 1

1 (a) Find the derivatives of the following distributions:

(i) $\chi_{[1,2]}(x)$ (ii) $f(x) = \begin{cases} -x, & x < 0 \\ 1, & 0 \le x \le 1 \\ x, & x > 1 \end{cases}$

(b) For the following distribution T and function f decide whether $T(f)$ makes sense. If so, evaluate it. If not, say why.

(i) $T(x) = \delta'(x)$, $f(x) = \sin(2x)$

(ii) $T(x) = \chi_{[0,\infty)}(x)$, $f(x) = \frac{1}{1+|x|}$

(iii) $T(x) = \left(x^2\delta(x-1)\right)'$, $f(x) = 1$.

(c) By using the properties of convolutions, show that for a continuous function g and C^1 function f

$$\frac{d}{dt}\left(g * (f\chi_{[0,\infty)})\right) = g * (f'\chi_{[0,\infty)}) + f(0)g.$$

2 (a) Show that

$$u(t) = \left(e^{-t} - e^{-2t}\right)\chi_{[0,\infty)}(t)$$

is a fundamental solution for the operator $D^2 + 3D + 2$.

(b) Use your answer to part (a) to find a general solution to

$$y''(t) + 3y'(t) + 2y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{1+t^2}, & t \ge 0. \end{cases}$$

Your answer should contain an integral, but try to simplify this as much as possible - in particular see if you can remove any characteristic functions by adjusting the limits.

(c) If the initial conditions $y(0) = 0$ and $y'(0) = 0$ must also be satisfied, there is a unique solution. What is it? (The result of 1c may be very useful here.)

(d) Plot the solution from part (c) for $t = 0..10$.

You should plot this using MAPLE and then either print it out, or sketch your solution. You'll need to use the convolution procedure in the downloadable MAPLE worksheet. To plot a convolution of functions $f(x)$, $g(x)$ try code of the following type:

> F:=x-> `convolve`(x->f(x), x->g(x), x); (this creates a function $F(x) = (f * g)(x)$)
> `plot`(F(x), x=0..10,thickness=2); (this plots F from x = 0 to 10)

Notes: (1) to compute a convolution of the form $(\chi_{[0,\infty)}f) * (\chi_{[0,\infty)}g)$ it is computational more efficient to use the algorithm `Lapconvolve` rather than `convolve`, but the syntax is the same.

(2) The ; at the end of each line of MAPLE code are necessary, but can be replaced by : to suppress output.

(3) To run an execution group, click on the group and hit return. To add a new line of text rather than running the code, hold down shift and hit return.

(4) The code `chi`(a,b) yields the function $\chi_{[a,b]}$ in MAPLE. For ∞ write out `infinity`. Alternatively the function $\chi_{[0,\infty)}$ can be written `Heaviside`. (In this special case, the inbuilt MAPLE command is usually preferable)

(5) The MAPLE syntax for e^x is `exp`(x)

(6) To enter a product in MAPLE you must use *. For example $2x$ is written 2*x. For powers use ˆ, e.g. x^2 is written xˆ2.

(7) The `Lapconvolve`, `convolve` and `chi` are all user written programs, MAPLE won't recognise them unless you run the execution group containing them first. If you want to use them in another worksheet, you'll need to copy and paste them across.