DISTRIBUTIONS AND FUNCTION SPACES.

ABSTRACT. This is a reference guide to the basic definitions and properties of distributions.

1. FUNCTION SPACES

Throughout our discussion of differential equations we shall need to make different technical assumptions about the functions under consideration. In this section, we group together the definitions of the relevant function spaces.

Definition 1.1.

- *The set $C^k(\mathbb{R})$ consists of all real-valued functions $f(x)$ on the real line such that all derivatives of order $\leq k$ exist and each derivative $f^{(j)}(x)$ $j \leq k$ is continuous on all of $\mathbb{R}$. We use $C^\infty(\mathbb{R})$ for functions with continuous derivatives of all orders. When $k = 0$ we shall normally just write $C(\mathbb{R})$ instead of $C^0(\mathbb{R})$.*
- *The set $C_0^k(\mathbb{R})$ consists of the C^k functions that vanish outside some finite interval $[a, b]$. Such functions are said to have compact support.*
- *The set $L^p(\mathbb{R})$, $p \geq 1$ consists of all (integrable) real-valued functions $f(x)$ such that*
$$\int_{-\infty}^{\infty} |f(x)|^p dx < \infty.$$
For a function $f \in L^p(\mathbb{R})$ we define the L^p-norm of f to be
$$\|f\|_{L^p(\mathbb{R})} = \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{1/p}.$$
(We won't discuss here exactly what integrable means, but any piecewise continuous function is integrable.)

Example 1.2.

(a) Consider $f(x) = \begin{cases} 0, & x < 0 \\ x^2, & x \geq 0 \end{cases}$. Then we can compute that
$$f'(x) = \begin{cases} 0, & x < 0 \\ 2x, & x \geq 0 \end{cases}, \qquad f''(x) = \begin{cases} 0, & x < 0 \\ 2, & x > 0 \end{cases}$$
Thus f is in $C^1(\mathbb{R})$ but not $C^2(\mathbb{R})$. Now f does not vanish for $x > b$ for any $b \in \mathbb{R}$ so f does not have compact support.

(b) The function $f(x) = \begin{cases} 0, & x < -1 \\ (x-1)^2(x+1)^2, & -1 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$ has compact support as it vanishes outside $[-1, 1]$. It is also in $C^1(\mathbb{R})$, so $f \in C_0^1(\mathbb{R})$.

(c) For any subset $E \subset \mathbb{R}$, the characteristic function of E is the function $\chi_E = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$. The characteristic function of any set $E \neq \mathbb{R}, \emptyset$ is not continuous, but if $E = [a, b]$ a finite interval, then $\chi_E \in L^p(\mathbb{R})$ for all $p \geq 1$.

(d) The function $f(x) = \dfrac{\chi_{[1,\infty]}(x)}{x}$ is in $L^2(\mathbb{R})$ but not $L^1(\mathbb{R})$. Why?

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_1^{\infty} \frac{1}{x^2} dx = -\frac{1}{x}\Big|_1^{\infty} = 1 < \infty$$

but

$$\int_{-\infty}^{\infty} |f(x)|^1 dx = \int_1^{\infty} \frac{1}{x} dx = \ln x\Big|_1^{\infty} = \infty.$$

(e) All functions $f \in C_0^k(\mathbb{R})$ for $k \geq 0$ are contained in $L^p(\mathbb{R})$ for all $p \geq 1$. Why? The function $f(x)$ has contact support, so is only non-zero on some interval $[a, b]$. But $f \in C([a, b])$ so it must be bounded above and below on $[a, b]$. Thus there is some $K > 0$ such that $-K \leq f(x) \leq K$ everywhere. This implies that $|f(x)|^p \leq K^p$ for all x and so we see

$$\|f\|_{L^p(\mathbb{R})} \leq (b - a)^{1/p} K$$

and f must be in $L^p(\mathbb{R})$.

□

Note: The next lot of definitions are technically necessary for some of the theory, but will largely live behind the scenes. Don't get too hung up on trying to understand them immediately.

Definition 1.3. *These function spaces each have their own notion of convergence.*

- *A sequence of (f_n) of L^p functions L^p-converges to $f \in L^p$ iff* [1]

$$\|f - f_n\|_{L^p(\mathbb{R})} \to 0 \quad \textit{as } n \to \infty.$$

- *A sequence (f_n) of C_0^k functions C^k-converges to $f \in C_0^k$ iff there is some finite interval $[a, b]$ such that all of the f_n and f vanish outside $[a, b]$ and*

$$\max_{x \in [a,b]} |f_n(x) - f(x)| \to 0 \quad \textit{as } n \to \infty.$$

(There is a notion of convergence for C^k not just C_0^k but it is more difficult to state and work with.)

Example 1.4.

(a) For any $f \in L^1(\mathbb{R})$, e.g. $f(x) = \frac{1}{1+x^2}$, the sequence of function $f_n(x) = f(x)\chi_{[-n,n]}(x)$ converge to $f(x)$ in L^1. For out example

$$\int_n^{\infty} \frac{1}{1 + x^2} dx = \frac{\pi}{2} - \tan^{-1}(n) \to 0 \quad \text{as } n \to \infty.$$

A similar statement is true for $\int_{-\infty}^{-n}$ so $\|f - f_n\|_{L^1(\mathbb{R})} \to 0$.

(b) It's hard to show, but its true that for any $f \in L^p(\mathbb{R})$ there is a sequence of functions $f_n \in C_0^\infty(\mathbb{R})$ such that f_n L^p-converges to f.

□

[1]iff means "if and only if"

2. Distributions

The space $C_0^\infty(\mathbb{R})$ of functions that have continuous derivatives of all orders and vanish outside some finite interval $[a, b]$ have another name. They are often called *test functions.*

Test functions behave incredibly well when integrated and differentiated. In particular,

$$f \in C_0^\infty(\mathbb{R}) \Longrightarrow f' \in C_0^\infty(\mathbb{R}) \tag{1}$$

i.e. the set of test functions is stable under differentiation. (Other useful spaces have this stability property: one frequently used in differential equations is the Schwartz class, p.87 in Logan.)

Furthermore, they have the following key property: if $f \in C^1(\mathbb{R})$ and ϕ is a test function then $\phi f'$ is continuous with compact support and hence in $L^1(\mathbb{R})$. Integration by parts then yields

$$\int_{-\infty}^{\infty} f'(x)\phi(x)dx = -\int_{-\infty}^{\infty} f(x)\phi'(x). \tag{2}$$

In other words, we can move the derivative onto the test function. As we shall soon see, this will allow us to generalize both the idea of a function and the idea of a derivative.

Definition 2.1. *A distribution T is a map from $C_0^\infty(\mathbb{R})$ to $\mathbb{R}$, (i.e. a procedure that takes a test function as input and spits out a real number) that satisfies*

- *$T(\phi + \psi) = T(\phi) + T(\psi)$ for all test functions ϕ, ψ.*
- *$T(c\phi) = cT(\phi)$ for all constants c and test functions ϕ.*
- *If ϕ_n converges to ϕ in C_0^∞ then $T(\phi_n) \to T(\phi)$.*

Note: this third convergence property is technically necessary, but in the cases we shall consider is always true and will be pushed into the background.

Example 2.2.

(a) Every piecewise continuous function f can be thought of as a distribution by considering f to be the equal to the map T_f defined by

$$T_f(\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)dx$$

(Because ϕ is a test function this integral always exists and is finite)

(b) Caution: different functions can be the same when thought of as distributions, e.g. the Heaviside function $\chi_{[0,\infty)}$ and $\chi_{(0,\infty)}$ are the same as distributions. Fortunately different **continuous** functions yield different distributions.

(c) Not every distribution comes from a function. The inaptly named Dirac delta "function" at p is defined by

$$\delta_p(\phi) = \phi(p).$$

This is a distribution that cannot be expressed as T_f for any function f. The delta "function" can be thought of as an infinitely thin, infinitely high spike with area 1.

(d) **Products**: If f is a C^∞ function and T is a distribution, we can define the product fT as a distribution by

$$(fT)(\phi) = T(f\phi).$$

Unless one is the distribution associated to a function, the product of two distributions is generally meaningless!

□

Remark 2.3. *Distributions formally only make sense when applied to a test function. However, specific examples can be applied to a wider range of functions. For example, the delta function δ_0 makes sense when applied to anything that is continuous near 0. We will often apply distributions to more general functions whenever everything still makes sense.*

Similarly we can often define products of functions not in C^∞ with distributions. To define fT all we require is that $T(f\phi)$ make sense whenever $\phi \in C_0^\infty(\mathbb{R})$.

The integration by parts formula (2) allows us to extend the notion of derivatives to distributions.

Definition 2.4. *For a distribution T we define its (weak) derivative T' to be the distribution*

$$T'(\phi) = -T(\phi').$$

(Here its useful to note that the derivative of a test function is again a test function.)

Example 2.5.

(a) If f is a C^1 function then the weak distribution derivative agrees with the usual derivative, i.e. $T_f' = T_{f'}$. This follows immediately from (2).

(b) The derivative of the Heaviside function $\chi_{[0,\infty)}$ is the delta "function" at 0. Why?

$$\int_{-\infty}^{\infty} \chi_{[0,\infty)}(x)\phi'(x)dx = \int_0^\infty \phi'(x) = \phi(x)\Big|_0^\infty = -\phi(0) = -\delta_0(\phi).$$

(The top limit vanishes for the last step because all test functions vanish outside some finite interval.)

(c) **Product Rule:** Suppose T is a distribution and f is C^∞ function. Then

$$(fT)'(\phi) = T'(f\phi) + T(f'\phi).$$

For example,

$$\begin{aligned}(e^x\chi_{[0,\infty)})'(\phi) &= \delta_0(e^x\phi) + \int_{-\infty}^{\infty} \chi_{[0,\infty)}(x)e^x\phi(x)dx \\ &= e^0\phi(0) + \int_{-\infty}^{\infty} \chi_{[0,\infty)}(x)e^x\phi(x)dx.\end{aligned}$$

Or alternatively

$$(e^x\chi_{[0,\infty)})' = \delta_0 + e^x\chi_{[0,\infty)}.$$

□