

1. Find the solution $u = u(x, t)$ of the heat equation

$$4u_{xx} = u_t, \quad 0 < x < \pi, \quad t > 0$$

which satisfies the boundary conditions

$$u(0, t) = 0 = u(\pi, t), \quad t \geq 0$$

and the initial condition

$$u(x, 0) = \pi - x, \quad 0 < x < \pi.$$

2. Use the complex exponential function to show that

$$\cos(x + y) = \cos x \cos y - \sin x \sin y.$$

3. Discuss the Gibbs' phenomenon for the function

$$f(x) = \begin{cases} -1, & \text{if } -\pi \leq x < 0 \\ 1, & \text{if } 0 \leq x \leq \pi \end{cases}$$

near its discontinuity 0 by working through these steps.

- (a) Show that the Fourier series of f is

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} = \frac{4}{\pi} \left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right).$$

- (b) Show that

$$\sum_{n=1}^N \cos(2n-1)x = \frac{\sin 2Nx}{2 \sin x}.$$

- (c) Find the first few critical numbers to the right of the origin of the error function

$$E_N(x) = \frac{4}{\pi} \sum_{n=1}^N \frac{\sin(2n-1)x}{2n-1} - f(x).$$

- (d) With x equal to the critical numbers you found in part (c), find

$$\lim_{N \rightarrow \infty} E_N(x),$$

thereby approximating the amount that the N th Fourier partial sums, for N large, over and undershoot the function f near 0. Your answers may contain definite integrals.

4. From class and homework, we know that the solution to the heat conduction problem

$$\begin{aligned}\alpha^2 u_{xx} &= u_t, & 0 < x < \pi, & \quad t > 0 \\ u(0, t) &= 0 = u(\pi, t), & t &\geq 0 \\ u(x, 0) &= f(x), & 0 < x < \pi.\end{aligned}$$

is

$$u(x, t) = \sum_{k=1}^{\infty} b_k e^{-\alpha^2 k^2 t} \sin kx$$

where

$$b_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin kx \, dx.$$

The goal of this problem is for you to show that if t_0 is any positive number, then all the derivatives of the function $h(x) = u(x, t_0)$ exist at all points x in the interval $(0, \pi)$, even if the initial temperature function $f(x)$ fails to be differentiable, or even continuous, at some points in $(0, \pi)$.

- Find the n th derivative $\frac{d^n}{dx^n} u(x, t_0)$ formally by differentiating the terms of the series for $u(x, t_0)$.
- Using the fact that all the Fourier coefficients b_k satisfy $|b_k| \leq M$ for some constant M , show that the k th term of the series you found in part (a) by formal differentiation has absolute value $\leq M k^n e^{-\alpha^2 k^2 t_0}$.
- Show that the series

$$\sum_{k=1}^{\infty} M k^n e^{-\alpha^2 k^2 t_0}$$

converges. (Hint: Use the ratio test which you studied when working with Taylor series. The ratio test is discussed in any Calculus book if you can't remember it.) It now follows that the series you found in part (a) converges at every point x in the interval $(0, \pi)$. Say why if you can. In addition, it follows (using some ideas from Math 63 which you are not responsible for) that the sum of the series you found in part (a) is $\frac{d^n}{dx^n} u(x, t_0)$.

5. Let

$$f(x) = \begin{cases} 0, & \text{if } -\pi \leq x < 0 \\ \pi - x, & \text{if } 0 \leq x \leq \pi \end{cases}$$

- Find the exponential form of the Fourier series of f by evaluating the integrals giving the coefficients of the series.
- By grouping the terms in the series of part (a) appropriately, find the trigonometric form of the Fourier series of f . (You can check your work by observing that any function g can be written as the sum of an even function and an odd function as follows:

$$g(x) = \frac{g(x) + g(-x)}{2} + \frac{g(x) - g(-x)}{2}.$$

Both the Fourier series of the even and odd functions into which f decomposes in this way have come up in this course.)