Math 31 – Proofs Practice

The following problems are for proof-writing practice and will not be submitted for grade.

Miscellaneous:

1. Find a closed formula for the sum of the first n odd numbers
$$\sum_{i=1}^{n}(2i-1).$$
Prove it by induction.

2. Let $x \in \mathbb{Q}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$. Prove by contradiction that $x+y \notin \mathbb{Q}$ (the sum of a rational number and an irrational number is irrational. What does it mean for a number to be rational?)

3. For integers a and b, $a \neq 0$, we say that a *divides* b, written $a|b$, if there exists some integer d such that $b = ad$ (ie. b is divisible by a). Let $a, b \in \mathbb{Z}_{\neq 0}$ be such that a divides b and b divides a. Prove that $b \in \{a, -a\}$.

Functions and Set Theory:

1. A function $f : A \to B$ is *injective* if for all $x, y \in A$, if $f(x) = f(y)$, then $x = y$ (no two elements are mapped to the same place under f. A function $f : A \to B$ is *surjective* if for all $y \in B$, there is some $x \in A$ such that $f(x) = y$ (every element in B is "hit" by something in A.

   Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Is f injective? Is it surjective? Prove your answer.

2. Let A and B be two sets. Prove (by considering set elements) that $A \cup B = A$ if and only if $B \subseteq A$. (Whenever you see *if and only if*, it means there are two things to prove!)

3. (De Morgan's Law) Let $X, Y \subset U$. Show that $(X \cup Y)^c = X^c \cap Y^c$. Here, X^c denotes the set complement $U \setminus X$. (What does it mean for two sets to be equal?)

Linear Algebra:

1. Prove (using associativity of multiplication) that if A is an invertible matrix, its inverse is unique.

2. Prove that if $S, T : \mathbb{R}^n \to \mathbb{R}^n$ are linear functions, then $S \circ T$ is a linear function. (Recall that T is linear if for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, $T(c\mathbf{v}) = cT(\mathbf{v})$ and $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$.

3. Prove by induction that if $\mathbf{v}$ is an eigenvector for a square matrix A with eigenvalue λ, then for all positive integers n, $\mathbf{v}$ is an also eigenvector for A^n with eigenvalue λ^n.