Math 31 Take Home Final

Due August 28, 2018

Instructions: This is an open notes, open book exam. You may use your notes or homework assignments from class, as well as Pinter 2nd edition. You may not use the internet or any other textbooks, nor may you confer with classmates. If you use any unauthorized aid, this is an honor code violation. This exam is due at 4pm on August 28.

1. Let H and K be normal subgroups of a group G such that $H \cap K = \{e\}$. Prove that G is isomorphic to a subgroup of $G/H \times G/K$.

2. (a) Prove that any group of order 4 is abelian.

   (b) Let G be a group and $Z(G)$ be the center of G. Show that if $G/Z(G)$ is cyclic, then G is abelian.

   (c) Let $[G : Z(G)] = 4$. What is the isomorphism class of $G/Z(G)$?

3. Let G be an abelian group of order 16 with elements $a, b \in G$ such that the order of both a and b is 4, but $a^2 \neq b^2$. What is the isomorphism class of G?

4. Show that $\mathbb{Z}$ is a principal ideal domain.

5. Let $I = \langle 2 \rangle$ be an ideal of $\mathbb{Z}$.

   (a) Is I a maximal ideal of $\mathbb{Z}$?

   (b) Is $I[x]$ an ideal of $\mathbb{Z}[x]$? Is it maximal?

6. Suppose that a and b are elements of a commutative ring R such that ab is a zero-divisor. Show that either a is a zero-divisor or b is a zero-divisor.

7. Let R be a ring with m elements. Show that the characteristic of R divides m.

8. Let $f(x) = x^3 + 6 \in \mathbb{Z}_7[x]$. Write $f(x)$ as a product of irreducible polynomials.

9. Let R be an integral domain. Prove that if every proper ideal is a prime ideal, then R is a field. (*Hint:* Consider the ideal $\langle a^2 \rangle$ for an element $a \in R$).