NAME: GRADE:

Quiz 2

Rings, ideals and Arithmetic

1. Let $A, B \in \mathbb{R}[X]$ be polynomials and consider the rational function $f(X) = \dfrac{A(X)}{B(X)}$. Prove that there exist polynomials $\alpha, \beta \in \mathbb{R}[X]$ such that:

$$f(X) = \alpha(X) + \frac{\beta(X)}{B(X)} \qquad \text{and} \qquad d^\circ(\beta) < d^\circ(B).$$

2. Let A be a commutative ring with identity 1_A and J an ideal in A.

It is well-known that A/J is a commutative ring with identity (do **not** prove it).

a. Recall **without justification** what $0_{A/J}$ and $1_{A/J}$ are.

b. Let $x \in A$. Verify that the set $J_x = \{ax + j \; ; \; a \in A \, , \; j \in J\}$ is an ideal in A.

c. Prove that if J is a maximal ideal, then A/J is a field.

3. Let A and B be rings, I an ideal in A and $\varphi \in \mathrm{Hom}(A, B)$ a ring homomorphism.

Find a necessary and sufficient condition on φ for the function

$$\begin{aligned} \tilde{\varphi} : A/I &\longrightarrow B \\ [a] &\longmapsto \varphi(a) \end{aligned}$$

to be well-defined.

4. Let A be a commutative ring. Recall that a proper ideal I in A is said *prime* if for $a, b \in A$,

$$ab \in I \quad \Rightarrow \quad a \in I \text{ or } b \in I.$$

a. Determine all the prime ideals in $\mathbb{Z}$.

b. Assume that in the ring A, every ideal is of the form $\langle a \rangle = aA$ for some $a \in A$. Prove that in such a ring, prime ideals are maximal.

Reminder (do **not** prove this): an intersection of ideals is an ideal.

c. Describe the ideal $\langle 4 \rangle \cap \langle 6 \rangle$ of $\mathbb{Z}$.

d. Let $m, n \in \mathbb{Z}$. Describe the ideal $\langle m \rangle \cap \langle n \rangle$ of $\mathbb{Z}$.