Quiz 1

Composition laws, groups and subgroups

Solution

1. Consider the composition law defined on $\mathbb{Z}$ by

$$p * q = p - 2q.$$

Is it associative?

Let p, q and r be integers. Then

$$\begin{aligned} (p * q) * r &= (p - 2q) * r \\ &= p - 2q - 2r \end{aligned}$$

while

$$\begin{aligned} p * (q * r) &= p * (q - 2r) \\ &= p - 2(q - 2r) \\ &= p - 2q + 4r. \end{aligned}$$

The expressions seem different in general so we should be able to supply a counterexample to the associativity property for the law $*$.

Indeed, if $p = q = 0$ and $r = 1$, we get

$$(0 * 0) * 1 = -2 \neq 4 = 0 * (0 * 1).$$

Therefore, $*$ is not associative.

2. The composition law defined on $\Gamma = \{(x, y) \in \mathbb{R} \times \mathbb{R}\ ,\ y \neq 0\}$ **by**

$$(a, b) \boxtimes (c, d) = (ad + bc, bd)$$

is associative. Is $(\Gamma, \boxtimes)$ **a group?**

First, we look for a neutral element, that is a couple (e_1, e_2) such that, for every $(a, b) \in \Gamma$,

$$(a, b) \boxtimes (e_1, e_2) = (e_1, e_2) \boxtimes (a, b) = (a, b).$$

In particular e_1 and e_2 must be such that

$$ae_2 + be_1 = a \qquad \text{and} \qquad be_2 = b$$

for all $(a, b) \in \Gamma$. The second condition implies that $e_2 = 1$ and the first condition then becomes

$$a + be_1 = a.$$

Since this must hold for any value of a and b, then necessarily $e_1 = 0$.

At this point, we have proved that *if* there is a neutral element then it must be $(0, 1)$. Conversely, a direct computation shows that

$$(a, b) \boxtimes (0, 1) = (0, 1) \boxtimes (a, b) = (a, b)$$

for every $(a, b) \in \Gamma$, which proves that $(0, 1)$ is neutral for $\boxtimes$.

Next, we study the existence of inverses. For $(a, b) \in \Gamma$, the equation

$$(a, b) \boxtimes (x, y) = (0, 1)$$

is equivalent to

$$ay + bx = 0 \qquad \text{and} \qquad by = 1.$$

It follows from the second equation that $y = \frac{1}{b}$ and the first equation yields

$$\frac{a}{b} + bx = 0$$

so that $x = -\dfrac{a}{b^2}$. Again, one verifies that

$$(a, b) \boxtimes (-\frac{a}{b^2}, \frac{1}{b}) = (-\frac{a}{b^2}, \frac{1}{b}) \boxtimes (a, b) = (0, 1)$$

for all $(a, b) \in \Gamma$, so that every element in Γ has an inverse.

Conclusion: $(\Gamma, \boxtimes)$ is a group.

3. Let G be a group with neutral element e and a, b elements in G satisfying

$$a^{-1}ba^{-1} = b^{-1}ab^{-1}.$$

Solve simultaneously the equations $ax^2 = b$ and $x^3 = e$.

Composing with a^{-1} on the left, the first equation leads to

$$x^2 = a^{-1}b.$$

Composing on the left with x^{-1} then gives

$$x^3 = xa^{-1}b$$

and the second equation implies that

$$xa^{-1}b = e.$$

In other words, if x is solution of the problem, it must be the inverse of $a^{-1}b$ in G:

$$x = b^{-1}a.$$

Let us verify that $x = b^{-1}a$ actually satisfies both equations. First,

$$ax^2 = ab^{-1}ab^{-1}a.$$

Notice that the assumption on a and b can be rephrased as $ab^{-1}ab^{-1}a = b$ (compose with a on the left and on the right) so that the first equation is satisfied.

Finally,

$$\begin{aligned} x^3 &= b^{-1}ab^{-1}ab^{-1}a \\ &= (b^{-1}ab^{-1})(a^{-1}ba^{-1})^{-1} \\ &= (b^{-1}ab^{-1})(b^{-1}ab^{-1})^{-1} \qquad \text{by the relation on } a \text{ and } b \\ &= e. \end{aligned}$$

This shows that the problem admits a unique solution, namely $x = b^{-1}a$.

4. Let $H = \left\{ \dfrac{p}{2^n} \; , \; p \in \mathbb{Z} \; , \; p \neq 0 \; , \; n \in \mathbb{N} \right\}$**. Is** H **a subgroup of** $(\mathbb{Q}^\times, \times)$**?**

Although it is stable under products, we observe that H is not stable under inverses. For instance $\dfrac{3}{2}$ belongs to H but its inverse in $\mathbb{Q}^\times$ is $\dfrac{2}{3}$, which cannot be written with a power of 2 as its denominator.

Therefore, H is not a subgroup of $\mathbb{Q}^\times$.