NAME: GRADE:

1. Let G and G' be groups and $m : G \longrightarrow G'$ a group homomorphism.
Prove that $\ker m$ is a subgroup of G.

Note: we proved in class that kernels are normal subgroups. **Do not use this fact.**

By definition, $\ker(m) = \{g \in G \mid m(g) = e_{G'}\} \subset G$

- Since $m \in \mathrm{Hom}(G, G')$, $m(e_G) = e_{G'}$ so $e_G \in \ker(m)$.
- Let $x, y \in \ker(m)$.

Then $m(x^{-1}y) = m(x)^{-1} m(y)$ since $m \in \mathrm{Hom}(G, G')$

$= e_{G'}^{-1} e_{G'}$

$= e_{G'}$

Therefore, $x^{-1}y \in \ker(m)$.

By the Subgroup Criterion, ker (m) is a subgroup of G.

2. Let G be an abelian group. Prove that every subgroup of G is normal.

Let H be a subgroup of G.

For $h \in H$ and $g \in G$, $\quad ghg^{-1} = gg^{-1}h = h \in H$

so $\underline{H \triangleleft G}$

Since H is arbitrary, this shows that every subgroup of an abelian group is normal.

3. Consider the additive group $C^\infty(\mathbb{R})$ of differentiable functions on $\mathbb{R}$ and the map $\varphi : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ defined by

$$\varphi(f) = f' - f.$$

a. Is φ a group homomorphism?

Let $f, g \in C^\infty(\mathbb{R})$. Then
$$\begin{aligned}\varphi(f+g) &= (f+g)' - (f+g)\\ &= f' + g' - f - g\\ &= \underbrace{f' - f} + \underbrace{g' - g}\\ &= \varphi(f) + \varphi(g)\end{aligned}$$

Therefore, $\underline{\varphi \in \mathrm{Hom}(C^\infty(\mathbb{R}), C^\infty(\mathbb{R}))}$.

b. Is φ injective?

Let us study the kernel of φ: $\varphi(f) = 0 \iff f = f'$.

Since the function e^x satisfies this equation, it is a non-zero element of $\ker(\varphi)$.

Therefore, $\underline{\ker(\varphi) \text{ is not injective}}$.

4. Recall that a matrix $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with real coefficients is invertible if and only if $ad - bc \neq 0$, in which case its inverse is

$$g^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The multiplicative group of invertible matrices of size 2×2 is denoted by $\mathrm{GL}(2, \mathbb{R})$.

a. Consider the map $T : \mathrm{GL}(2, \mathbb{R}) \longrightarrow \mathbb{R}$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d.$$

Is T group homomorphism?

The neutral elements of $GL(2, \mathbb{R})$ and $\mathbb{R}$ are respectively

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and 0.

Since $T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2 \neq 0$, T is not a homomorphism.

b. Prove that the subset H of matrices of the form $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ with $a > 0$ and $b \in \mathbb{R}$ is a subgroup of $\mathrm{GL}(2, \mathbb{R})$.

- Every matrix in H is invertible since $a \cdot 1 - b \cdot 0 = a > 0$

  so $H \subset GL(2, \mathbb{R})$.

- The subset H is not empty: it contains $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $(a=1, b=0)$.

- It is stable under product: for $a, \alpha > 0$ and $b, \beta \in \mathbb{R}$,

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a\alpha & a\beta + b \\ 0 & 1 \end{bmatrix}$$

  and $a\alpha > 0$ as the product of two positive numbers.

- Finally, H is stable under inverse: if $a > 0$ and $b \in \mathbb{R}$,

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{a} \begin{bmatrix} 1 & -b \\ 0 & a \end{bmatrix} = \begin{bmatrix} 1/a & -b/a \\ 0 & 1 \end{bmatrix}$$

  with $\frac{1}{a} > 0$.

Therefore, $\underline{H \text{ is a subgroup of } GL(2, \mathbb{R})}$

c. Prove that the map $\psi : H \longrightarrow \mathrm{GL}(2,\mathbb{R})$ defined by

$$\psi\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ \ln(a) & 1 \end{bmatrix}$$

is a group homomorphism.

Let $a, \alpha > 0$ and $b, \beta \in \mathbb{R}$.

$$\psi\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix}\right) = \psi\left(\begin{bmatrix} a\alpha & a\beta+b \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ \ln(a\alpha) & 1 \end{bmatrix}$$

On the other hand,

$$\psi\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right)\psi\left(\begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ \ln(a) & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \ln(\alpha) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \ln(a)+\ln(\alpha) & 1 \end{bmatrix}$$

Since $\ln(a\alpha) = \ln(a) + \ln(\alpha)$, it follows that $\psi \in \mathrm{Hom}(H, GL(2,\mathbb{R}))$

d. Determine $\ker \psi$.

Let $h = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in H$.

$$\psi(h) = \begin{bmatrix} 1 & 0 \\ \ln(a) & 1 \end{bmatrix}$$

so $\psi(h) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \iff a = 1$

Therefore, $\ker(\psi) = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, b \in \mathbb{R} \right\}$.

5. Let G be a group and H, K subgroups of G. We assume that the reunion $H \cup K$ is a subgroup of G.

Prove that $H \subset K$ or $K \subset H$.

Assume that $H \not\subset K$. Then, there exists $h \in H$ such

that $h \notin K$.

To prove that $K \subset H$, let $k \in K$.

Since $h, k \in H \cup K$, which is a group, $hk \in H \cup K$

Therefore, either $hk \in H$ or $hk \in K$

If $hk \in K$, then $h = \underbrace{hk}_{\in K} \underbrace{k^{-1}}_{\in K} \in K$

this would contradict the definition of h !

Therefore, $hk \in H$ and it follows that

$$k = \underbrace{h^{-1}}_{\in H} \underbrace{hk}_{\in H} \in H.$$

We have proved that every element of k also belongs to H,

that is $K \subset H$ $\blacksquare$

6. Let G be a group. Its *center* is by definition the set $Z(G)$ of elements that commute with all the elements of G:

$$Z(G) = \{g \in G\,,\ ag = ga \quad \text{for all } a \in G\}\,.$$

a. Prove that the relation $\mathcal{R}$ defined on G by

$$x\mathcal{R}y \quad \Leftrightarrow \quad \exists g \in G\,,\ y = gxg^{-1}$$

is an equivalence relation.

Reflexivity: $\forall x \in G$, $x = e_G x e_G^{-1}$ so $x R x$.

Symmetry: If $y = gxg^{-1}$, then $x = g^{-1}y(g^{-1})^{-1}$ so $xRy \Rightarrow yRx$

Transitivity: Assume that $y = gxg^{-1}$ and $z = hyh^{-1}$.

then $z = hgxg^{-1}h^{-1} = hg\,x\,(hg)^{-1}$ so $z R x$.

We have proved that conjugacy is an equivalence relation

b. Let $z \in Z(G)$. Determine the equivalence class of z.

Assume that $y R z$. Then, there exists some $g \in G$ such that

$y = gzg^{-1}$. Since $z \in Z(G)$, this implies that

$y = gg^{-1}z = z$.

Therefore $[z] = \{z\}$.

c. Prove that $Z(G)$ is a subgroup of G.

The center of G is included in G by definition and $e_G \in Z(G)$.

If $x, y \in Z(G)$ and $a \in G$, then

$$x^{-1} y a \underset{\substack{\uparrow \\ y \in Z(G)}}{=} x^{-1} a y = (a^{-1} x)^{-1} y \underset{\substack{\uparrow \\ x \in Z(G)}}{=} (x a^{-1})^{-1} y = a x^{-1} y$$

Therefore, by the Subgroup Criterion, $\underline{Z(G) < G}$.

d. Is $Z(G)$ normal in G?

Let $z \in Z(G)$ and $g \in G$.

Then $g z g^{-1} = z g g^{-1} = z \in Z(G)$

so $\underline{Z(G) \triangleleft G}$

7. Let G be a group with neutral element e and a, b elements in G such that

$$a^2 = e \quad , \quad b^3 = e \quad , \quad ab = ba.$$

Let Γ be the subgroup of G generated by $\{a, b\}$.

a. Prove that Γ is necessarily abelian.

The elements of Γ are finite words in a^k and b^ℓ with $k, \ell \in \mathbb{Z}$.
Since a and b commute, so do their powers. Therefore, every element of Γ is of the form $a^k b^\ell$ with $k, \ell \in \mathbb{Z}$.

If $\gamma_1 = a^{k_1} b^{\ell_1}$ and $\gamma_2 = a^{k_2} b^{\ell_2}$ are elements in Γ, then

$$\gamma_1 \gamma_2 = a^{k_1} b^{\ell_1} a^{k_2} b^{\ell_2} = a^{k_1 + k_2} b^{\ell_1 + \ell_2}$$
$$= a^{k_2} b^{\ell_2} a^{k_1} b^{\ell_1} = \gamma_2 \gamma_1$$

Therefore, Γ is abelian.

b. How many different elements can Γ contain (at the most)?

The conditions $a^2 = e$ and $b^3 = e$ respectively imply that

$$\{a^k, k \in \mathbb{Z}\} = \{a^0 = e, a^1 = a = a^{-1}\}$$
$$\{b^\ell, \ell \in \mathbb{Z}\} = \{b^0 = e, b^1 = b, b^2 = b^{-1}\}$$

Therefore, $\Gamma = \{e, a, b, b^2, ab, ab^2\}$

so Γ has at most six elements.