NAME: GRADE:

Math 31

Midterm Examination

Rules

- This is a **closed book exam**. No document is allowed.
- Cell phones and other electronic devices must be turned off.
- Questions and requests for clarification can be addressed to the instructor only.
- You are allowed to use the result of a previous question even if you did not prove it, as long as you indicate it explicitly.

Grading

- In order to receive full credit, solutions must be **justified with full sentences**.
- The clarity of your explanations will enter the appreciation of your work.
- Every individual question is worth 5 points.

Last piece of advice

Read the entire exam before you start to write anything.

1. Let G and G' be groups and $m : G \longrightarrow G'$ a group homomorphism.
Prove that $\ker m$ is a subgroup of G.

Note: we proved in class that kernels are normal subgroups. **Do not use this fact.**

2. Let G be an abelian group. Prove that every subgroup of G is normal.

3. Consider the additive group $\mathcal{C}^\infty(\mathbb{R})$ of differentiable functions on $\mathbb{R}$ and the map $\varphi : \mathcal{C}^\infty(\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R})$ defined by

$$\varphi(f) = f' - f.$$

a. Is φ a group homomorphism?

b. Is φ injective?

4. Recall that a matrix $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with real coefficients is invertible if and only if $ad - bc \neq 0$, in which case its inverse is

$$g^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The multiplicative group of invertible matrices of size 2×2 is denoted by $\mathrm{GL}(2, \mathbb{R})$.

a. Consider the map $T : \mathrm{GL}(2, \mathbb{R}) \longrightarrow \mathbb{R}$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + d.$$

Is T a group homomorphism?

b. Prove that the subset H of matrices of the form $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ with $a > 0$ and $b \in \mathbb{R}$ is a subgroup of $\mathrm{GL}(2, \mathbb{R})$.

c. Prove that the map $\psi : H \longrightarrow \mathrm{GL}(2, \mathbb{R})$ defined by

$$\psi\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ \ln(a) & 1 \end{bmatrix}$$

is a group homomorphism.

d. Determine $\ker \psi$.

5. Let G be a group and H, K subgroups of G. We assume that the reunion $H \cup K$ is a subgroup of G.

Prove that $H \subset K$ or $K \subset H$.

6. Let G be a group. Its *center* is by definition the set $Z(G)$ of elements that commute with all the elements of G:

$$Z(G) = \{g \in G\,,\ ag = ga \quad \text{for all } a \in G\}\,.$$

a. Prove that the relation $\mathcal{R}$ defined on G by

$$x\mathcal{R}y \quad \Leftrightarrow \quad \exists g \in G\,,\ y = gxg^{-1}$$

is an equivalence relation.

b. Let $z \in Z(G)$. Determine the equivalence class of z.

c. Prove that $Z(G)$ is a subgroup of G.

d. Is $Z(G)$ normal in G?

7. Let G be a group with neutral element e and a, b elements in G such that

$$a^2 = e \qquad , \qquad b^3 = e \qquad , \qquad ab = ba.$$

Let Γ be the subgroup of G generated by $\{a, b\}$.

a. Prove that Γ is necessarily abelian.

b. How many different elements can Γ contain (at the most)?