Math 31
Final Examination

due August 29, 2016

Write complete justifications to get full credit.

Problem 1. Cauchy's Theorem for abelian groups and application

Let G be an abelian group with order $n \geq 2$.

a. Let g be an element of order m in G. Prove that for every positive divisor d of m, there exists an element of G with order d.

The purpose of this problem is to prove by induction on n that if p is a prime divisor of n, then G contains at least one element of order p.

b. Check that the result holds for $n \in \{2, 3, 4\}$.

Now we fix n and assume that the property is satisfied for every group of order $< n$.

c. Let $\gamma \in G \setminus \{e_G\}$ be an element of order relatively prime with p, a prime divisor of n. Prove that $G/\langle\gamma\rangle$ contains an element of order p.

d. Deduce that G contains an element of order p.

From now on, let G be an abelian group of order 10.

e. Prove that G contains an element τ of order 2 and an element σ of order 5.

f. Construct an isomorphism between $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ and G.

g. Is G cyclic?

Problem 2. Field extensions

The questions in this problem are mutually independent.

a. Find a monic irreducible polynomial P in $\mathbb{Q}[X]$ such that $\mathbb{Q}[1+\sqrt{2}] \simeq \mathbb{Q}[X]/\langle P\rangle$.

b. Let $F = \mathbb{Z}/2\mathbb{Z}$. Prove that $F[X]/\left\langle X^3+X+1\right\rangle$ is a field and construct its addition and multiplication tables.

c. Let K be a subfield of a field F. Prove that the elements of F that are algebraic over K form a subfield of F.

Problem 3. Isomorphism of two quotient rings

The goal of this problem is to determine whether the rings

$$A = \mathbb{R}[X]/\left\langle X^2-1\right\rangle \qquad \text{and} \qquad B = \mathbb{R}[X]/\left\langle X^2-2X+1\right\rangle$$

are isomorphic or not.

a. Verify that the map

$$\begin{array}{rcl} \varphi : \mathbb{R}[X] & \longrightarrow & \mathbb{R}[X]/\langle X-1\rangle \times \mathbb{R}[X]/\langle X+1\rangle \\ P & \longmapsto & (P+\langle X-1\rangle, P+\langle X+1\rangle) \end{array}$$

is a ring homomorphism.

b. Prove that A is isomorphic to $\mathbb{R}[X]/\langle X-1\rangle \times \mathbb{R}[X]/\langle X+1\rangle$.

c. Prove that B contains an element $b \neq 0$ such that $b^2 = 0$.

d. Are A and B fields? Are they isomorphic rings?

Problem 4. Is $\mathbb{Z}[X]$ a Euclidean ring?

a. Describe the smallest ideal I of $\mathbb{Z}[X]$ that contains 2 and X.

b. Is I principal?

c. Does the Euclidean Division Theorem apply in $\mathbb{Z}[X]$?

Problem 5. Final question

Read up on Evariste Galois and Niels Abel. Suggest actors to play their role in Hollywood biopics (not just based on their looks).