Reducibility over $\mathbb{Q}$ implies reducibility over $\mathbb{Z}$

Let $f(x) \in \mathbb{Z}[x]$ and $f(x) = g(x)h(x)$ such that $g(x), h(x) \in \mathbb{Q}[x]$. We need to show that it is possible to find $g'(x), h'(x) \in \mathbb{Z}[x]$ such that $f(x) = g'(x)h'(x)$.

Step 1: It is possible to find a positive integer c and integer polynomials $g'(x), h'(x) \in \mathbb{Z}[x]$ such that $cf(x) = g'(x)h'(x)$.

Indeed, let a be the least common multiple of the denominators of all coefficients from $g(x)$ and b be such least common multiple for $h(x)$. Take $g'(x) = ag(x)$ and $h'(x) = bh(x)$. Then $g'(x), h'(x) \in \mathbb{Z}[x]$.

Example: if $g(x) = 3x^3 - \frac{2}{3}x^2 + \frac{7}{5}x + \frac{1}{2}$, then $a = 30$ and $g'(x) = 30g(x) = 90x^3 - 20x^2 + 42x + 15$.

Now, $abf(x) = (ag(x))(bh(x)) = g'(x)h'(x)$, so let's take $c = ab$. This finishes **Step 1**.

Step 2: Choose c to be the smallest positive number that can be used in **Step 1** (we know that at least one such number exists, so there must be the smallest one). Our goal is to show that $c = 1$.

Assume that $c > 1$. Let p be some prime divisor of c. Let $\overline{g'}(x), \overline{h'}(x) \in \mathbb{Z}_p[x]$ be obtained from $g'(x)$ and $h'(x)$ by reducing all their coefficients modulo p. Once again, we know that

$$cf(x) = g'(x)h'(x)$$

Then (since $c \bmod p = 0$):

$$0 = (cf(x)) \bmod p = g'(x)h'(x) \bmod p = \overline{g'}(x)\overline{h'}(x)$$

Since $\mathbb{Z}_p[x]$ is an integral domain (see Fact 1 from the handout about Polynomial rings), either $\overline{g'}$ or $\overline{h'}$ **must** be zero. We can assume that $\overline{g'}(x) = 0$. This means that all coefficients of $g'(x)$ are divisible by p. Take $c' = c/p \in \mathbb{Z}$ and $g''(x) = g'(x)/p \in \mathbb{Z}[x]$. Then

$$c'f(x) = g''(x)h'(x)$$

Since $c' < c$, this is a **contradiction** to the condition on c to be the smallest one, so our assumption that $c > 1$ is wrong.

Conclusion. It follows that the minimal positive value of c from **Step 1** is 1 and, hence, $f(x) = g'(x)h'(x)$ for some $g'(x), h'(x) \in \mathbb{Z}[x]$.