Math 31 Winter 2005

Topics in Algebra

Final Exam (take-home)

due at 1:55PM on Wednesday March 16
in the Instructor's office 402 Bradley Hall

Your name (please print): ________________________________

Instructions: This is a open book, open notes exam. You can use any printed matter (or your class notes) of your choice but you **can not** consult one another or other humans. **Use of calculators is not permitted**. You must justify all of your answers to receive credit.

The instructor will be in his office on Wednesday March 16 **from noon till 2PM** to collect the exam. If he is not there when you submit your exam, please write the **submission time** on this front page and slide the exam under the office door.

The exam total score is the sum of your **10** (out of **11**) best scores. Please do all your work on the paper provided.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only

1.	________	/15
2.	________	/15
3.	________	/15
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6.	________	/15
7.	________	/15
8.	________	/15
9.	________	/15
10.	________	/15
11.	________	/15
Total:	________	/150

1. Are $2\mathbb{Z}/12\mathbb{Z}$ and $\mathbb{Z}_6$ isomorphic as groups? Are they isomorphic as rings? Justify your answer.

2. Prove that every subgroup of D_n of **odd** order is cyclic. **Hint:** prove first that no such subgroup can contain any reflections.

3. Let G and H be groups such that $G \oplus H$ is cyclic. Prove that both G and H are cyclic.

4. Prove that for each integer $n > 0$ the factor group $\mathbb{Q}/\mathbb{Z}$ has a unique subgroup of order n.

5. Let G be a (not necessarily Abelian) group of order 108 and let $\varphi : G \to \mathbb{Z}_{18}$ be a homomorphism that is **onto**. Prove that G has a normal subgroup of order 6.

6. Let F be a field with $125 = 5^3$ elements. Prove that the mapping $\varphi : F \to F$ that sends $x \in F$ into x^5 is a ring homomorphism.

7. Find the characteristic of $\mathbb{Z}_n \oplus \mathbb{Z}_m$. Justify your answer.

8. Let $R = \mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$ be a ring and $I = \langle\sqrt{3}\rangle = \sqrt{3}R$ be its principal ideal generated by $\sqrt{3}$.

a Find the number of elements in the factor ring R/I;

b Prove that I is a maximal ideal of R.

9. Prove that every ring homomorphism $\varphi : \mathbb{Z}_n \to \mathbb{Z}_n$ has the form $\varphi(x) = ax$, where $a^2 = a$. For which a is φ a ring isomorphism?

10. For which $a \in \mathbb{Z}_8$ does the polynomial $4x+a$ have a multiplicative inverse in $\mathbb{Z}_8[x]$? Justify your answer.

11. Prove that $x^4 + 1$ is irreducible over $\mathbb{Q}$ but reducible over $\mathbb{R}$.