Main Properties of External Direct Products

Theorem 1. **(Order of an Element in a Direct Product)**

Let $G_1, G_2, \ldots, G_n$ be groups, $g_i \in G_i$, and $(g_1, g_2, \ldots, g_n) \in G_1 \oplus G_2 \oplus \cdots \oplus G_n$.
Then $|(g_1, g_2, \ldots, g_n)| = \text{lcm}(|g_1|, |g_2|, \ldots, |g_n|)$.

Theorem 2. **(When is $G \oplus H$ Cyclic)**

Let G and H be finite cyclic groups. **Then** $G \oplus H$ is cyclic **if and only if** $|G|$ and $|H|$ are relatively prime.

Corollary 2.1. **(When is $G_1 \oplus G_2 \oplus \cdots \oplus G_n$ Cyclic)**

An external direct product $G_1 \oplus G_2 \oplus \cdots \oplus G_n$ of finite cyclic groups is cyclic **if and only if** $|G_i|$ and $|G_j|$ are relatively prime for all $i \neq j$.

Corollary 2.2. **(When $Z_{n_1 n_2 \cdots n_k} \approx \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$)**

Let $m = n_1 n_2 \cdots n_k$. **Then** $\mathbb{Z}_m$ is isomorphic to $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ **if and only if** n_i and n_j are relatively prime for all $i \neq j$.

Theorem 3. **($U(n)$ as and External Direct Product)**

Let s and t be relatively prime. **Then**

a. $U(st) \approx U(s) \oplus U(t)$;
b. $U_s(st) \approx U(t)$;
c. $U_t(st) \approx U(s)$.

Corollary 3.1. Let $m = n_1 n_2 \cdots n_k$ such that n_i and n_j are relatively prime for all $i \neq j$.
Then $U(m) \approx U(n_1) \oplus U(n_2) \oplus \cdots \oplus U(n_k)$.