NAME: ______________________________

Solution

Math 31

Midterm Examination

Rules

- This is a **closed book exam**. No document is allowed.
- Cell phones and other electronic devices must be turned off.
- Questions and requests for clarification can be addressed to the instructor only.
- You are allowed to use the result of a previous question even if you did not prove it, as long as you indicate it explicitly.

Grading

- In order to receive full credit, solutions must be **justified with full sentences**.
- The clarity of your explanations will enter into the appreciation of your work.

Last piece of advice

Read the entire exam before you start to write anything.

Problem	1	2	3	4	5	6	7	Total
Points	6	7	8	7	6	8	8	50
Score								

1. *(6 points)* Let $*$ be the operation on $\mathbb{R}$ given by

$$x * y = (x - y)^4$$

Explain whether or not

i) the operation is commutative,

ii) there is an identity element e with respect to $*$,

iii) if for every element there is an inverse with respect to $*$.

i) $*$ is commutative:

$$x * y = (x-y)^4 = (x-y)^2 (x-y)^2$$
$$y * x = (y-x)^4 = (y-x)^2 (y-x)^2$$

ii) there is no identity element e for the operation: For $x < 0$ we have $(x-e)^4 \geq 0$, but $x < 0$

Hence $x * e = (x-e)^4 \neq x$

iii) As there is no identity element, there are no inverses.

2. *(7 points)* Let $G = \{e, a, b, c\}$ be a set of four elements, where e denotes the neutral element. Using an operation table, find all possible groups with these four elements, where

$$b \cdot c = a.$$

Justify your answer.

$*$	e	a	b	c
e	e	a	b	c
a	a			
b	b			a
c	c			

We know that our operation table is like on the left.

Rule: In every row and column each element occurs exactly once.

Case: 1.) $a * c = b \overset{\text{Rule}}{\Longrightarrow} c * c = e$

a) $a * a = e \overset{\text{Rule}}{\Longrightarrow} a * b = c$

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	$\boxed{c}$	e	a
c	c	b	a	e

$\overset{\text{Rule}}{\Longrightarrow} b * a = c$

$\rightsquigarrow$ 1 table

$\boxed{\cong \mathbb{Z}_2 \times \mathbb{Z}_2}$

b) $a * a = c \Rightarrow a * b = e$

$*$	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	$\boxed{e}$	c	a
c	c	b	a	e

$\overset{\text{Rule}}{\Longrightarrow} b * a = e$

$\rightsquigarrow$ 1 table

$\boxed{\cong \mathbb{Z}_4}$

2.) $a * c = e \overset{\text{Rule}}{\Longrightarrow} c * c = b$

a) $a * a = b \overset{\text{Rule}}{\Longrightarrow} a * b = c$

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	$\boxed{e}$	a
c	c	e	a	b

$\overset{\text{Rule}}{\Longrightarrow} b * b = e$

$\rightsquigarrow$ 1 table

$\boxed{\cong \mathbb{Z}_4}$

b) $a * a = c \overset{\text{Rule}}{\Longrightarrow} a * b = b$

This is not possible, as $e * b = b$

A contradiction to our rule

3. *(8 points)* Let $(G, \cdot)$ be a group.

a. Is $f : G \to G, x \mapsto f(x) = x^2$ a bijective function for any group G? Justify your answer.

No: Take $G = (\mathbb{R} \setminus \{0\}, \cdot)$

Then $f(x) = x^2 = y^2 = f(y) \Rightarrow x = y$ or $x = -y$

b. Show that for fixed $a \in G$ the function $h : G \to G, x \mapsto h(x) = a^3 x a^{-3}$ is a bijective function.

a) h is injective: $h(x) = h(y)$

$$\Leftrightarrow a^3(xa^{-3}) = a^3(ya^{-3})$$

$$\overset{\text{Cancellation}}{\Longrightarrow} xa^{-3} = ya^{-3} \overset{\text{Cancellation}}{\Longrightarrow} x = y$$

b) h is surjective: Take $y \in G$. Then

$$h(x) = y \Leftrightarrow a^3 x a^{-3} = y$$

$$\Rightarrow x = a^{-3} y a^3$$

So for every y there is $x = a^{-3} y a^3$, s.th. $h(x) = y$

c. Is the function $h : G \to G$ from part **b.** a group isomorphism? Justify your answer.

Yes, h is an isomorphism. We check:

for all x, y in G:

h is a homomorphism

$$\left\{\begin{array}{l} h(x \cdot y) = a^3 x y a^{-3} \\ h(x) \cdot h(y) = (a^3 x a^{-3})(a^3 y a^{-3}) = a^3 x y a^{-3} \\ \text{so } h(x \cdot y) = h(x) \cdot h(y) \quad \forall x, y \in G \end{array}\right.$$

as h is bijective and a homomorphism we know that h is an isomorphism.

4. *(7 points)* For each of the following statements, either **prove** that it is true **or** explain why it is **false**.

a. Let $(G, \cdot)$ be an arbitrary group. Then

$$H = \{x \in G, \text{ such that } x = y^2 \text{ for some } y \in G\} \quad \text{is a subgroup of } G.$$

$\boxed{\text{False!}}$ If $x = y^2$ and $a = b^2$

Then in general: $xa = y^2 b^2 \neq (yb)^2$

Hence the third criterion for a subgroup does not have to be satisfied

b. Let $(G, \cdot)$ be an arbitrary group. Suppose that K and M are subgroups of G. Then

$$K \cdot M = \{x \cdot y, x \in K \text{ and } y \in M\} \quad \text{is a subgroup of } G.$$

$\boxed{\text{False}}$ Again if $x \cdot y, ab \in K \cdot M$

Then $(xy)(ab)$ does not have to be in $K \cdot M$ if G is not abelian

c. If every element of a group $(G, \cdot)$ is its own inverse, then G is abelian.

$\boxed{\text{True!}}$ If $a = a^{-1}$ and $b = b^{-1}$

Then $ab = a^{-1}b^{-1} = (ba)^{-1} = ba$

This is true for all $a, b \in G$

Hence G is abelian

5. *(6 points)* Let $M_2(\mathbb{R})$ be the set of 2×2 matrices with real coefficients. Show that

$$A \sim B \Leftrightarrow B = P \cdot A \cdot P^{-1} \text{ for some } P \in GL_2(\mathbb{R})$$

a. Show that $\sim$ is an equivalence relation in $M_2(\mathbb{R})$.

I) $A \sim A$: as $Id \cdot A \cdot Id^{-1} = A$

II) $A \sim B \Rightarrow B \sim A$:
if $A \sim B$, then $PAP^{-1} = B$ for some $P \in GL_2(\mathbb{R})$

$\Rightarrow A = P^{-1}BP = P^{-1}B(P^{-1})^{-1} \Rightarrow B \sim A$

III) $A \sim B$ and $B \sim C \Rightarrow A \sim C$

As $A \sim B$ we know $B = PAP^{-1}$ for some P
$B \sim C$ " " $C = QBQ^{-1}$ for some Q

$\Rightarrow C = (QP)A(P^{-1}Q^{-1}) = (QP)A(QP)^{-1}$

$\Rightarrow C \sim A$

b. Write down the equivalence class of

$$[Id] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \text{ of the matrix } Id.$$

$[Id] = \{B \in M_2(\mathbb{R}) \mid Id \sim B\}$

$= \{B \in M_2(\mathbb{R}) \mid B = \underbrace{P^{-1} Id P}_{= Id} \text{ for some } P\}$

$= \{Id\}$

So this class contains only Id

6. *(8 points)* Consider the group $(\mathbb{Z} \times \mathbb{Z}_2, + \times +_2)$.

a. Describe or list all elements of $\mathbb{Z} \times \mathbb{Z}_2$.

$$\mathbb{Z} \times \mathbb{Z}_2 = \{(k,1), k \in \mathbb{Z}\} \cup \{(k,0), k \in \mathbb{Z}\}$$

b. Draw the Cayley graph

$$\Gamma_1 = \Gamma(\mathbb{Z} \times \mathbb{Z}_2, \{(2,0),(3,0),(0,1)\})$$

of $\mathbb{Z} \times \mathbb{Z}_2$ with respect to the generating set $\{(2,0),(3,0),(0,1)\}$.

(2,0)

(3,0)

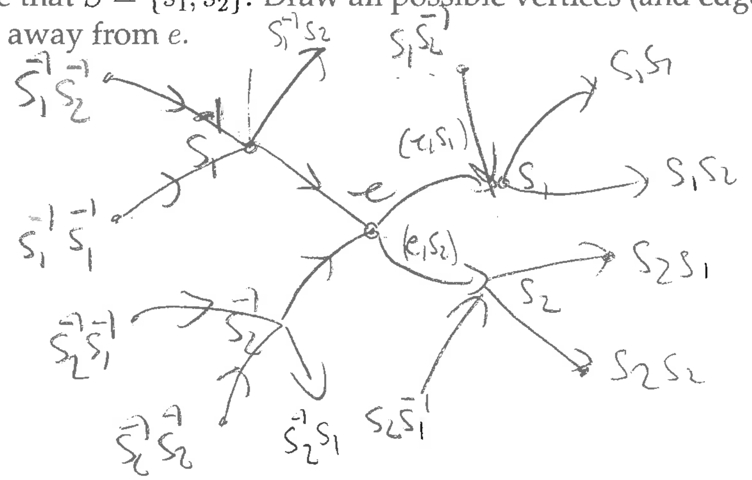
(0,1)

7. *(8 points)* Let $(G, \cdot)$ be a group with neutral element e and let S be a generating set i.e. $\langle S \rangle = G$. We recall that

$$G = \langle S \rangle = \{s_1 \cdot s_2 \cdot \ldots \cdot s_n, \text{ where } s_i \in S \cup S^{-1} \text{ and } n \in \mathbb{N}\}.$$

Let $\Gamma = \Gamma(G, S)$ be the corresponding Cayley graph of G with respect to S.

a. Suppose that $S = \{s_1, s_2\}$. Draw all possible vertices (and edges) that are at most two edges away from e.

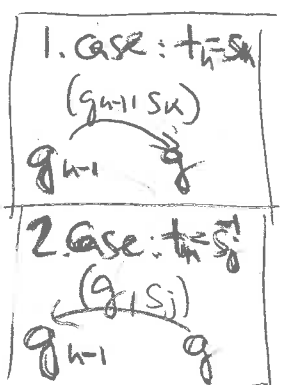


b. We now look at the general case. Suppose that S has m elements. Show that every vertex $g \in G$ is connected to the vertex $e \in G$ by a path of edges.

Given $g = t_1 \cdot t_2 \cdots t_n$ where $t_i \in S \cup S^{-1}$

Then $\boxed{e}$ is connected to $\boxed{t_1}$ by an edge
where $\boxed{e}$ is either starting or endpoint
$\boxed{t_1}$ is connected to $\boxed{t_1 t_2}$ by an edge
where $\boxed{t_1}$ is either starting or end point

⋮



$\boxed{g_{n-1} = t_1 \cdots t_{n-1}}$ is connected to $\boxed{t_1 \cdots t_n}$ by an edge
where $\boxed{t_1 \cdots t_{n-1}}$ is either starting or endpoint.

In total $\boxed{e}$ is connected to $\boxed{g = t_1 \cdots t_n}$.